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TRANSFORM ANALYSIS FOR POINT PROCESSES AND APPLICATIONS IN CREDIT RISK

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This paper develops a formula for a transform of a vector point process with totally inaccessible arrivals. The transform is expressed in terms of a Laplace transform under an equivalent probability measure of the point process compensator. The Laplace transform of the compensator can be calculated explicitly for a wide range of model specifications, because it is analogous to the value of a simple security. The transform formula extends the computational tractability offered by extant security pricing models to a point process and its applications, which include valuation and risk management problems arising in single-name and portfolio credit risk.

KEY WORDS: point process, transform, compensator, equivalent measure change.

1. INTRODUCTION

A point process is a stochastic model for the arrival of events. In finance, point processes are used to describe the timing of corporate and sovereign defaults, the arrival of orders in a limit order book, and the timing of jumps in asset prices. This paper develops a formula for a transform of a point process. The transform facilitates the treatment of a range of applications, including the valuation and risk management of securities exposed to the credit risk of a single or multiple firms, the prediction of order arrivals in a limit order book, and the measurement of risk in asset portfolios.

To outline our results, let L be a vector of d point processes with totally inaccessible arrivals and positive and independent and identically distributed jump amplitudes, r be a stochastic discount rate process, and Y be a nonnegative random variable that may represent a cash flow at a horizon T. The component point processes of L may be correlated, and both r and Y may depend on L. For $w, z \geq 0, u \in \mathbb{R}^d_+$ and $t \leq T$, we consider the point process transform

$$\mathbb{E}\left[\exp\left(-w\int_t^T r_s ds\right) e^{-zY-u\cdot(L_T-L_t)} \,\middle|\, \mathcal{F}_t\right]. \tag{1.1}$$

For $w = z = 0$, (1.1) represents the Laplace transform of $L_T - L_t$, which encodes the conditional distribution of L. More generally, for suitable functions h, (1.1) can be

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inverted to obtain an expression for the "conditional expected payoff"

$$\mathbb{E}\left[\exp\left(-w\int_t^T r_s ds\right) h(Y, L_T)\,\middle|\,\mathcal{F}_t\right]. \tag{1.2}$$

To appreciate the significance of the transform (1.1) for credit risk applications, suppose the point process L describes the losses in d portfolios of defaultable assets such as loans or corporate bonds. Then the expectation (1.2) could represent the price of a credit derivative referenced on these portfolios such as a credit or tranche swap, the conditional distribution of total losses at T, or other portfolio risk measures. By exploiting the multivariate formulation and letting some of the components of L represent the losses due to counterparty default, we can address the implications for derivatives valuation of counterparty risk. Moreover, the dependence structure between the loss events of the components of L, the discount rate r, and the payoff variable Y is not constrained, allowing for the modeling of the correlation between defaults within and across portfolios, the flight-to-quality phenomena observed during the crisis of 2007–2009, the dependence between default and recovery rates documented by Altman et al. (2005), as well as self- and cross-exciting effects. The latter effects may be associated with contagion, by which the default by one name has an impact on the conditional default rates of other names, within and beyond a portfolio. Azizpour, Giesecke, and Schwenkler (2010) provide strong evidence of the presence of contagion effects for US corporate defaults, after controlling for the exposure of firms to observable and frailty risk factors.

We show that the point process transform (1.1) is given by $\mathcal{L}_t^u(\psi(u), z, w, T)$, where $\psi(u)$ is a d-vector of functions $\psi_n(u_n) = 1 - \int \exp(-u_n x) F^n(dx)$ with F^n the law of the jump amplitudes of the component point process L^n, and where

$$\mathcal{L}_t^u(v, z, w, T) = \mathbb{E}^u\left[\exp\left(-w\int_t^T r_s ds\right) e^{-zY - v\cdot(A_T - A_t)}\,\middle|\,\mathcal{F}_t\right] \tag{1.3}$$

is a Laplace transform of the Doob–Meyer compensator A to the process counting the jumps of L. The Laplace transform (1.3) is taken under an equivalent probability measure $\mathbb{P}^u$ defined by the Radon–Nikodym derivative $\exp(\psi(u)\cdot A_T - u\cdot L_T)$. It is a familiar expression in finance. To see this, suppose the jumps of L arrive with intensity λ, i.e., $dA_t = \lambda_t dt$. Consider a security that pays $\exp(-zY)$ at time T. The price of this security is analogous to (1.3) once we regard the process $wr + v\cdot\lambda$ as the short rate. The explicit calculation of the security price is well understood for a large class of short-rate processes and payoffs Y; see Cheng and Scaillet (2007), Duffie, Pan, and Singleton (2000), Leippold and Wu (2002), and others. Our results extend this computational tractability to the point process transform (1.1) and its various applications.

The measure change underlying the Laplace transform (1.3) may not be necessary if L is a doubly stochastic Poisson process, i.e., if the jumps of the elements of L arrive according to independent Poisson processes given a realization of the compensator A. If this property is violated, in the presence of contagion effects, for example, or if the discount rate responds to events as with flight-to-quality effects, then the measure change "neutralizes" the self- and cross-excitation of the components of L. After the change of measure, L tends to jump less frequently because A is scaled down by a factor depending on the transform variable u. The larger the elements of u, the less frequent are the arrivals. As $u \to \infty$, the compensator vanishes after the change of measure.

Our equivalent change of probability measure is related to two other measure changes that have been developed in the credit risk and options pricing literatures. Collin-Dufresne, Goldstein, and Hugonnier (2004) introduce an absolutely continuous change of probability measure that facilitates the calculation of the probability $\mathbb{P}(L_T - L_t = 0 \mid \mathcal{F}_t)$ and of the prices of defaultable securities. The associated measure puts zero mass on paths for which events arrive before T. This absolutely continuous measure is obtained as the limit of our equivalent measure $\mathbb{P}^u$ as $u \to \infty$. Carr and Wu (2004) introduce a complex-valued measure change to develop a formula for the Fourier transform of a time-changed Lévy process, which is obtained by evaluating each of the elements of a d-dimensional Lévy process at a common stochastic clock. The associated complex measure is equivalent to the reference measure but not a probability measure. This complex measure formally coincides with the measure $\mathbb{P}^u$ we propose, if the Radon–Nikodym derivative is extended to the complex plane, and if the point process L can be represented as a time-changed Lévy process in the sense of Carr and Wu (2004). This representation exists only if the components of the compensator vector A are the same, i.e., if the events in each of the component processes of L share common arrival dynamics.

Our transform formula generalizes various existing formulae for the transforms of specific point processes. These include the formula developed by Ding, Giesecke, and Tomecek (2009) for a transform of a univariate point process obtained by time-changing a linear birth process with an independent process, the formula provided by Carr and Wu (2004) for a transform of a point process obtained by time-changing each of the components of a vector Poisson process with a common process that may depend on the Poisson process, and the formula developed by Errais, Giesecke, and Goldberg (2010) for a transform of a vector point process with affine structure. The transform formula we develop in this paper does not require any specific representation or structure.

Section 2 introduces the point process and our assumptions, and outlines the measure changes proposed by Collin-Dufresne, Goldstein, and Hugonnier (2004) and Carr and Wu (2004). Section 3 develops our equivalent change of probability measure. Section 4 provides the transform formula and discusses its computation. Section 5 provides an example. Section 6 shows how to use the transform to address several applications in single-name and portfolio credit risk. Section 7 concludes.

2. POINT PROCESSES AND MEASURE CHANGES

2.1. Point Processes

We fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a right-continuous and complete information filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. For $d \in \mathbb{N}$, let $N = (N^1, \ldots, N^d)$ be a vector of nonexplosive counting processes, each starting at 0. The totally inaccessible jump times $(T_i^n)_{i \geq 1}$ of N^n represent the ordered default stopping times in portfolio n. The default times may be correlated within a portfolio and across portfolios. However, we assume that there are no simultaneous defaults: $\Delta N_t^n \, \Delta N_t^m = 0$ almost surely for all t and $m \neq n$.

The Doob–Meyer theorem guarantees the existence of a process $A = (A^1, \ldots, A^d)$ such that $M = N - A$ is a local martingale. The compensator A has elements A^n that are nondecreasing and continuous almost surely. It is unique up to indistinguishability. Throughout, we fix a finite horizon $T > 0$ and suppose that for each n,

$$\mathbb{E}\left(\exp\left(A_T^n\right)\right) < \infty. \tag{2.1}$$

For each portfolio n, let $(\ell_i^n)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution F^n on $(0,\infty)$ satisfying $\int_0^\infty xF^n(dx) < \infty$. The variable $\ell_i^n \in \mathcal{F}_{T_i^n}$ represents the financial loss at the ith default in portfolio n. We require $\mathbb{P}(\ell_i^n \in dx|\mathcal{F}_{T_i^n-}) = F^n(dx)$ for all $i \geq 1$. The sequence (ℓ_i^n) is independent of (ℓ_i^m) for all $m \neq n$. The loss process $L = (L^1, \ldots, L^d)$ records the cumulative loss in each of the d portfolios:

$$L^n = \ell_1^n + \cdots + \ell_{N^n}^n. \tag{2.2}$$

We develop a measure change method for computing a transform of the point process L. The measure change we propose extends or complements two other measure changes that have appeared in the credit risk and options pricing literatures. To provide some perspective, we outline these other measure changes below.

2.2. Absolutely Continuous Measure Change

In an innovative paper, Collin-Dufresne, Goldstein, and Hugonnier (2004) introduce an absolutely continuous change of probability measure that facilitates the calculation of $\mathbb{P}(N_T^n = 0)$ and related quantities. They introduce a probability measure $\mathbb{P}_n^\infty$ on $\mathcal{F}_T$ defined by

$$\mathbb{P}_n^\infty(B) = \mathbb{E}(S_T^n 1_B), \quad B \in \mathcal{F}_T \tag{2.3}$$

with density

$$S_T^n = \frac{d\mathbb{P}_n^\infty}{d\mathbb{P}} = 1_{\{N_T^n=0\}} \exp(A_T^n). \tag{2.4}$$

The measure $\mathbb{P}_n^\infty$ is absolutely continuous with respect to $\mathbb{P}$. Because the probability $\mathbb{P}(S_T^n = 0) = \mathbb{P}(N_T^n > 0) > 0$, the measures $\mathbb{P}$ and $\mathbb{P}_n^\infty$ are not equivalent, however. Letting $\mathbb{E}_n^\infty$ denote expectation relative to $\mathbb{P}_n^\infty$, we nevertheless have

$$\begin{aligned} \mathbb{P}(N_T^n = 0) &= \mathbb{E}\big(S_T^n \exp\big(-A_T^n\big)\big) \\ &= \mathbb{E}_n^\infty\big(\exp\big(-A_T^n\big)\big). \end{aligned} \tag{2.5}$$

Formula (2.5) reduces the calculation of $\mathbb{P}(N_T^n = 0)$ to the calculation of the Laplace transform of the compensator A_T^n evaluated at 1, taken under the absolutely continuous measure $\mathbb{P}_n^\infty$. The measure $\mathbb{P}_n^\infty$ puts zero mass on paths for which N^n jumps by time T, i.e., paths with defaults in portfolio n prior to T. The Girsanov theorem for absolutely continuous measure changes developed by Lenglart (1977) prescribes the dynamics of A^n under $\mathbb{P}_n^\infty$, and facilitates the computation of the Laplace transform of A^n under $\mathbb{P}_n^\infty$. Only local $\mathbb{P}$-martingales V for which $[V, N^n] > 0$ will be adjusted when the measure is changed. In particular, the compensator of N^n under $\mathbb{P}_n^\infty$ is null on $[0, T]$.

To illustrate, let X be a risk factor solving $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dN_t^n$ for suitable functions μ and σ, where W is a standard Brownian motion and the compensator $A^n = \int_0^\cdot \Lambda^n(X_s)ds$ for a suitable function Λ^n. Since W is a $\mathbb{P}_n^\infty$-Brownian motion and the $\mathbb{P}_n^\infty$-compensator of N^n is null, X follows a diffusion $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$ on $[0, T]$ under $\mathbb{P}_n^\infty$. We see that the calculation of the Laplace transform $\mathbb{E}_n^\infty(\exp(-vA_T^n))$ for $v \geq 0$ is analogous to the calculation of the price of a nondefaultable zero bond paying 1 at T when the short rate follows $v\Lambda^n(X)$. This calculation is standard for a range of

specifications of (μ, σ, Λ^n), including affine and quadratic models. It is this property that renders formula (2.5) attractive from a computational perspective.

Formula (2.5) has applications in single-name credit risk. Take $\tau_n = \inf\{t : N_t^n > 0\}$ as the default time of name n. Then (2.5) provides a formula for the survival probability $\mathbb{P}(\tau_n > T) = \mathbb{P}(N_T^n = 0)$. This formula does not require the "no-jump hypothesis" of alternative survival probability formulae developed by Lando (1998) and others. The no-jump hypothesis is violated when N^n is self-exciting, as in the jump-diffusion intensity formulation discussed above. Here, A^n is influenced by N^n.

While mainly designed for a single-name setting, the absolutely continuous measure change has also applications in portfolio credit risk. Since the elements of N do not have common jumps, the product of the component densities

$$\begin{aligned} S_T &= S_T^1 \times \cdots \times S_T^d \\ &= 1_{\{N_T = 0_d\}} \exp(1_d \cdot A_T) \end{aligned} \tag{2.6}$$

defines an absolutely continuous probability measure

$$\mathbb{P}^\infty(B) = \mathbb{E}(S_T 1_B), \quad B \in \mathcal{F}_T. \tag{2.7}$$

Here and below, x_d denotes a d-vector with elements x. The measure $\mathbb{P}^\infty$ facilitates the computation of the joint survival probability:

$$\begin{aligned} \mathbb{P}(\tau_1 > T, \ldots, \tau_d > T) = \mathbb{P}(N_T = 0_d) &= \mathbb{E}(S_T \exp(-1_d \cdot A_T)) \\ &= \mathbb{E}^\infty(\exp(-1_d \cdot A_T)), \end{aligned} \tag{2.8}$$

where $\mathbb{E}^\infty$ denotes expectation relative to $\mathbb{P}^\infty$. Formula (2.8) provides an expression for the distribution of the first-to-default time $\min(\tau_1, \ldots, \tau_d)$, a quantity playing a key role in the valuation of basket derivatives; see Duffie (1998).

2.3. Complex-Valued Measure Change

In an insightful paper, Carr and Wu (2004) introduce a complex-valued measure change to develop a formula for the Fourier transform of a time-changed Lévy process $Y = (X_\theta^1, \ldots, X_\theta^d)$, which is obtained by evaluating each of the elements of a d-dimensional Lévy process $X = (X^1, \ldots, X^d)$ at a common stochastic clock $\theta = \int_0^\cdot \vartheta_s ds$, where $\vartheta_s \geq 0$. Carr and Wu (2004) consider a complex measure η^v on $\mathcal{F}_T$ defined by

$$\eta^v(B) = \mathbb{E}(U_T(v) 1_B), \quad B \in \mathcal{F}_T, \tag{2.9}$$

for $v \in \mathbb{R}^d$, with complex-valued density

$$U_T(v) = \exp(\Psi_X(v)\theta_T + iv \cdot Y_T), \tag{2.10}$$

where $i = \sqrt{-1}$, and where Ψ_X is the characteristic exponent of X. The measure η^v is equivalent to $\mathbb{P}$ but not a probability measure. Carr and Wu (2004) compute

$$\begin{aligned} \mathbb{E}(\exp(iv \cdot Y_T)) &= \mathbb{E}(U_T(v) \exp(-\Psi_X(v)\theta_T)) \\ &= \mathbb{E}_{\eta^v}(\exp(-\Psi_X(v)\theta_T)), \end{aligned} \tag{2.11}$$

where $\mathbb{E}_{\eta^v}(V)$ means $\int V(\omega) d\eta^v(\omega)$ for random variables V. The calculation of the Fourier transform of Y_T reduces to the calculation of a "Laplace transform"

$\mathbb{E}_{\eta^v}(\exp(-u\theta_T)), u \in \mathbb{C}$, of the time change $\theta_T = \int_0^T \vartheta_s ds$ under the complex measure η^v. Formally, this Laplace transform is analogous to the price of a nondefaultable bond paying 1 at T when the short-term rate of interest is $u\vartheta$. It is this analogy that renders formula (2.11) attractive, similarly to formula (2.5). Relative to the absolutely continuous measure change of Section 2.2, the complex-valued measure change generates additional issues that need to be addressed when calculating $\mathbb{E}_{\eta^v}(\exp(-u\theta_T))$ for a given specification of the time change θ. One such issue is the extension of the notion of a Brownian motion to the complex measure, and a corresponding Girsanov theorem.

The scope of the complex-valued measure change approach to the calculation of the Fourier transform also includes the point process $L = (L^1, \ldots, L^d)$ if there exists a Lévy process $X = (X^1, \ldots, X^d)$ and a scalar time change θ such that L can be constructed as $(X^1_\theta, \ldots, X^d_\theta)$. Such (X, θ) exist only in the special case that $A^1 = \cdots = A^d$, i.e., if all components of N have the same dynamics. In that case, we can take X as a standard compound Poisson process and $\theta = A^1$. Thus, the scope of the transform formula (2.11) includes only point processes with a particular structure.

3. EQUIVALENT CHANGE OF PROBABILITY MEASURE

We develop an equivalent change of probability measure that includes the absolutely continuous measure change (2.3) of Collin-Dufresne, Goldstein, and Hugonnier (2004) as a limiting case, and that is, in case $d = 1$, analogous to the complex-valued measure change (2.9) of Carr and Wu (2004). This measure change allows us to express a transform of L in terms of a Laplace transform of the compensator A. In our case, the Laplace transform of A is taken under an equivalent probability measure rather than a complex measure or a probability measure that is only absolutely continuous. This, in turn, facilitates its interpretation and computation. The resulting formula for the transform of L applies to any point process with continuous compensator satisfying (2.1).

PROPOSITION 3.1. *The stopped process* $(Z_{t\wedge T}(u))$ *defined for* $u \in \mathbb{R}^d_+$ *by*

$$Z_t(u) = \exp(\psi(u) \cdot A_t - u \cdot L_t), \tag{3.1}$$

where $\psi(u) = (\psi_1(u_1), \ldots, \psi_d(u_d))$ *and*

$$\psi_n(c) = 1 - \int_0^\infty e^{-cx} F^n(dx), \quad c \geq 0, \tag{3.2}$$

is a uniformly integrable martingale and induces an equivalent probability measure

$$\mathbb{P}^u(B) = \mathbb{E}(Z_T(u)1_B), \quad B \in \mathcal{F}_T. \tag{3.3}$$

Proof. Denote the random counting measure associated with the marked point process $(T^n_i, \ell^n_i)_{i\geq 1}$ by $\mu^n(dt, dx)$. Thus, $N^n_t = \int_0^t \int_0^\infty \mu^n(ds, dx)$ and $L^n_t =$

$\int_0^t \int_0^\infty x\mu^n(ds, dx)$. Since L^n is a pure jump process, for any $v \geq 0$ we have

$$\begin{aligned} e^{-vL_t^n} &= 1 + \sum_{0<s\leq t} \left(e^{-vL_s^n} - e^{-vL_{s-}^n}\right) \\ &= 1 + \int_0^t \int_0^\infty \left(e^{-v(L_{s-}^n + x)} - e^{-vL_{s-}^n}\right)\mu^n(ds, dx) \\ &= 1 + \int_0^t \int_0^\infty e^{-vL_{s-}^n}(e^{-vx} - 1)\mu^n(ds, dx). \end{aligned} \tag{3.4}$$

Since A^n is continuous,

$$\begin{aligned} e^{\psi_n(v)A_t^n} &= 1 + \int_0^t e^{\psi_n(v)A_s^n}\psi_n(v)\, dA_s^n \\ &= 1 - \int_0^t \int_0^\infty e^{\psi_n(v)A_{s-}^n}(e^{-vx} - 1)\nu^n(ds, dx), \end{aligned} \tag{3.5}$$

where $\nu^n(dt, dx)$ is the dual predictable projection of $\mu^n(dt, dx)$; see Dellacherie and Meyer (1982) for the general case and Jacod (1975) for the point process case. Since the marks satisfy $\mathbb{P}(\ell_i^n \in dx | \mathcal{F}_{T_i^n -}) = F^n(dx)$ for all $i \geq 1$, we have $\nu^n(dt, dx) = dA_t^n F^n(dx)$, where dA_t^n represents the random measure associated with the nondecreasing process A^n. By Stieltjes integration by parts and (3.4) and (3.5), we get

$$\begin{aligned} Z_t^n(v) &= e^{\psi_n(v)A_t^n - vL_t^n} \\ &= \int_0^t e^{\psi_n(v)A_{s-}^n} d\left(e^{-vL_s^n}\right) + \int_0^t e^{-vL_{s-}^n} d\left(e^{\psi_n(v)A_s^n}\right) + [e^{\psi_n(v)A^n}, e^{-vL^n}]_t \\ &= 1 + \int_0^t \int_0^\infty Z_{s-}^n(v)(e^{-vx} - 1)(\mu^n(ds, dx) - \nu^n(ds, dx)). \end{aligned} \tag{3.6}$$

Because

$$\int_0^t \int_0^\infty (\mu^n(ds, dx) - \nu^n(ds, dx))$$

defines a local martingale, (3.6) shows that $Z^n(v)$ is a local martingale.

Since A^n is nondecreasing, we see that for $v \geq 0$

$$\sup_{t\geq 0} \left| Z_{t\wedge T}^n(v) \right| \leq \sup_{t\geq 0} \exp\left(\psi_n(v) A_{t\wedge T}^n\right) \leq \exp\left(A_T^n\right).$$

It follows that $\sup_{t\geq 0} |Z_{t\wedge T}^n(v)|$ is integrable since $\exp(A_T^n)$ is, by condition (2.1). Then, by Theorem 51 in Chapter I of Protter (2004), $(Z_{t\wedge T}^n(v))$ is a uniformly integrable martingale, and so is the product $(Z_{t\wedge T}(u)) = (Z_{t\wedge T}^1(u_1) \cdots Z_{t\wedge T}^d(u_d))$, because the component processes of L do not have common jumps. □

There are two special cases of interest. Because $Z_T(0_d) = 1$, we have $\mathbb{P}^{0_d} = \mathbb{P}$, the reference probability measure. Since $\lim_{u\to\infty} \psi(u) = 1_d$ by dominated convergence, and the jumps of L are strictly positive almost surely, we also have

$$\lim_{u\to\infty} Z_T(u) = 1_{\{L_T = 0_d\}} \exp(1_d \cdot A_T) = S_T \tag{3.7}$$

almost surely, where S_T is given by the product (2.6) of the densities (2.4) inducing the absolutely continuous measures (2.3) of Collin-Dufresne, Goldstein, and Hugonnier

(2004). Thus we obtain the absolutely continuous measure $\mathbb{P}^\infty$ defined by S_T as the limiting measure

$$\lim_{u\to\infty} \mathbb{P}^u(B) = \mathbb{P}^\infty(B), \quad B \in \mathcal{F}_T. \tag{3.8}$$

The complex measure (2.9) of Carr and Wu (2004) can be obtained in the special case that $A^1 = \cdots = A^d$. In this case, L can be constructed by time-changing a compound Poisson process using a time change $\theta = A^1$, see Section 2.3. Extending the density $Z_T(u)$ defined by (3.1) to purely imaginary numbers by setting $u = -iv$, we see that $\mathbb{P}^u = \mathbb{P}^{-iv}$ is a complex measure that coincides with η^v for $v \in \mathbb{R}^d$.

4. POINT PROCESS TRANSFORM

The equivalent probability measure of Proposition 3.1 facilitates the computation of the point process transform

$$\varphi_t(u, z, w, T) = \mathbb{E}\left[\exp\left(-w\int_t^T r_s ds\right) e^{-zY-u\cdot(L_T-L_t)} \,\middle|\, \mathcal{F}_t\right] \tag{4.1}$$

where $w, z \geq 0, u \in \mathbb{R}^d_+, t \leq T, r$ is an adapted nonnegative and bounded discount rate process, and where $Y \in \mathcal{F}_T$ is a nonnegative random variable that represents a cash flow at T. Both r and Y may depend on L. The transform (4.1) encodes the distribution of L, the prices of securities exposed to L and Y, and other quantities of interest. Section 6 provides illustrative examples.

PROPOSITION 4.1. *The point process transform satisfies*

$$\varphi_t(u, z, w, T) = \mathcal{L}^u_t(\psi(u), z, w, T) \tag{4.2}$$

for $t \leq T, u \in \mathbb{R}^d_+$ *and* $z, w \geq 0$, *where for* $v \in \mathbb{R}^d_+$

$$\mathcal{L}^u_t(v, z, w, T) = \mathbb{E}^u\left[\exp\left(-w\int_t^T r_s ds\right) e^{-zY-v\cdot(A_T-A_t)} \,\middle|\, \mathcal{F}_t\right]. \tag{4.3}$$

Proof. By the definition of $Z(u)$, we have

$$\begin{aligned}\varphi_t(u, z, w, T) &= \mathbb{E}\left[\frac{Z_T(u)}{Z_t(u)} \exp\left(-w\int_t^T r_s ds\right) e^{-zY-\psi(u)\cdot(A_T-A_t)} \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}^u\left[\exp\left(-w\int_t^T r_s ds\right) e^{-zY-\psi(u)\cdot(A_T-A_t)} \,\middle|\, \mathcal{F}_t\right]\end{aligned}$$

and this yields the assertion. □

Formula (4.2) implies that the Laplace transform of a point process L with totally inaccessible arrivals is given by the Laplace transform of the counting process compensator A evaluated at ψ, the Laplace exponent of a compound Poisson process. The Laplace transform of A is taken under an equivalent probability measure $\mathbb{P}^u$. To calculate that transform, consider the specification of A in terms of a nonnegative intensity process

$\lambda = (\lambda^1, \ldots, \lambda^d)$ such that, for each n,

$$A_t^n = \int_0^t \lambda_s^n \, ds \tag{4.4}$$

almost surely. The process λ^n is the density of the random measure associated with A^n relative to Lebesgue measure. It represents the conditional mean arrival rate of defaults in portfolio n in the sense that $\mathbb{E}(N_{t+\Delta}^n - N_t^n \mid \mathcal{F}_t) \approx \lambda_t^n \Delta$ for "small" $\Delta > 0$. With the representation (4.4), the Laplace transform (4.3) takes the form

$$\mathcal{L}_t^u(v, z, w, T) = \mathbb{E}^u \left[\exp\left(- \int_t^T (w r_s + v \cdot \lambda_s) ds \right) e^{-zY} \,\middle|\, \mathcal{F}_t \right]. \tag{4.5}$$

The expectation on the right-hand side of equation (4.5) is a familiar expression in finance. Consider a security that pays $\exp(-zY)$ at time T. The price of this security at $t \leq T$ is analogous to formula (4.5) once we regard the process $(wr + v \cdot \lambda)$ as the short-term rate of interest. The explicit calculation of this price is well understood for a broad class of interest rate processes; see Cheng and Scaillet (2007), Duffie, Pan, and Singleton (2000), Leippold and Wu (2002) and others. Thus, Proposition 4.1 extends the analytical tractability offered by extant term structure models to the point process transform (4.1) and its applications, as discussed in Section 6 below.

The basic transform formula (4.2) can be extended in several directions. Using a multiparameter time index, we can compute the transform of a vector $(L_{T(1)}^1, \ldots, L_{T(d)}^d)$ of point processes evaluated at several time horizons $T(n)$. Another equivalent change of measure facilitates the computation of the transform of (L, N), where N counts the jumps of L. The associated Radon–Nikodym derivative involves a correction term that compensates the common jumps of L and N.

PROPOSITION 4.2. *On the interval* $[0, T]$, *the counting process* N^n *has* $\mathbb{P}^u$*-compensator* $(1 - \psi_n(u_n))A^n$ *and the* $\mathbb{P}^u$*-law of the loss magnitude* ℓ_1^n *is given by*

$$\mathbb{P}^u(\ell_1^n \in dx) = \frac{1}{1 - \psi_n(u_n)} \exp(-u_n x) F^n(dx). \tag{4.6}$$

Proof. Theorem 4.1 in Jacod (1975) implies that there exists a positive, finite, and predictable process $(\zeta^n(t, x))$ indexed by $x \in \mathbb{R}_+$ such that the random counting measure $\mu^n(dt, dx)$ associated with the marked point process (T_i^n, ℓ_i^n) has predictable projection under the equivalent measure $\mathbb{P}^u$ given by

$$\zeta^n(t, x) \nu^n(dt, dx), \tag{4.7}$$

where $\nu^n(dt, dx)$ is the predictable projection of $\mu^n(dt, dx)$ under $\mathbb{P}$. The process $(\zeta^n(t, x))$ can be identified from formula (14) in Jacod (1975), which implies an explicit expression for the Radon–Nikodym derivative process $Z = Z^1 \cdots Z^d$ in terms of the $\zeta^n(t, x)$ and the $\nu^n(dt, dx)$. Using the continuity of the A^n, the expression takes the form

$$\prod_{n=1}^d \prod_{i: T_i^n \leq t} \zeta^n(T_i^n, \ell_i^n) \exp\left(\int_0^t \int_0^\infty (1 - \zeta^n(s, x)) \nu^n(ds, dx) \right).$$

Using the fact that $\nu^n(dt, dx) = dA_t^n F^n(dx)$, we see that this expression is equal to $Z_t(u) = \exp(\psi(u) \cdot A_t - u \cdot L_t)$ for $u \in \mathbb{R}_+^d$ if and only if $\zeta^n(t, x) = \exp(-u_n x)$.

Returning to (4.7), we write the predictable projection of $\mu^n(dt, dx)$ under $\mathbb{P}^u$ as

$$\frac{\exp(-u_n x)}{1 - \psi_n(u_n)}(1 - \psi_n(u_n))\nu^n(dt, dx), \tag{4.8}$$

where $1 - \psi_n(u_n) = \mathbb{E}(\exp(-u_n \ell_1^n))$. The first factor in (4.8) represents the Radon-Nikodym derivative of the law of ℓ_1^n under $\mathbb{P}^u$ with respect to the law $F^n(dx)$ under $\mathbb{P}$; formula (4.6) follows immediately. The normalization guarantees that $\int_0^\infty \mathbb{P}^u(\ell_1^n \in dx) = 1$ for all u, as desired. The second factor $(1 - \psi_n(u_n))$ in (4.8) represents the Radon–Nikodym derivative of the random measure associated with the $\mathbb{P}^u$-compensator of N^n with respect to $dA_\cdot^n$. Thus the $\mathbb{P}^u$-compensator is given by $\int_0^\cdot (1 - \psi_n(u_n)) dA_s^n = (1 - \psi_n(u_n))A^n$. □

Proposition 4.2 facilitates the calculation of the Laplace transform (4.3) of the compensator A, which is taken under $\mathbb{P}^u$. While a $\mathbb{P}$-Brownian motion remains a Brownian motion under the measure $\mathbb{P}^u$ (as any other local martingale that does not have jumps in common with any of the N^n), the compensated local jump martingale $N - A$ does not remain a local martingale under $\mathbb{P}^u$. This means the compensator A must be adjusted when the measure is changed. Proposition 4.2 states that the compensator A^n to N^n is scaled by the $\mathbb{P}$-Laplace transform of the loss magnitude $1 - \psi_n(u_n) = \mathbb{E}(\exp(-u_n \ell_1^n))$ when the measure is changed to $\mathbb{P}^u$. Since $\mathbb{E}(\exp(-u_n \ell_1^n)) \leq 1$, N^n tends to jump less frequently under $\mathbb{P}^u$ than under $\mathbb{P}$. Under the absolutely continuous limiting measure $\mathbb{P}^\infty = \lim_{u \to \infty} \mathbb{P}^u$, the compensator vanishes on $[0, T]$, meaning that $\mathbb{P}^\infty(N_T = 0_d) = 1$. Note that also the law $F^n(dx)$ of the loss magnitude ℓ_1^n must be adjusted. From formula (4.6), the $\mathbb{P}^u$-Laplace transform of ℓ_1^n takes the form

$$\mathbb{E}^u\big(\exp\big(-v\ell_1^n\big)\big) = \frac{\mathbb{E}\big(\exp\big(-(v + u_n)\ell_1^n\big)\big)}{\mathbb{E}\big(\exp\big(-u_n\ell_1^n\big)\big)} = \frac{1 - \psi_n(v + u_n)}{1 - \psi_n(u_n)}, \quad v \geq 0. \tag{4.9}$$

The measure change may not be necessary when N is a doubly stochastic Poisson process. The process N is a doubly stochastic Poisson (or Cox) process if its compensator A is adapted to a right-continuous and complete sub-filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ of $\mathbb{F}$, and the increments $N_T^n - N_t^n$ are independent across n and have, given the sigma-field $\mathcal{F}_t \vee \mathcal{G}_T$, the Poisson distribution with parameter $A_T^n - A_t^n$; see Brémaud (1980):

$$\mathbb{P}(N_T - N_t = m \mid \mathcal{F}_t \vee \mathcal{G}_T) = \prod_{n=1}^d \frac{(A_T^n - A_t^n)^{m_n}}{m_n!} e^{-(A_T^n - A_t^n)} \tag{4.10}$$

for $t < T$ and $m = (m_1, \ldots, m_d) \in \mathbb{N}_0^d$. Equivalently, letting $\phi(u)$ be the d-vector of functions $(1 - e^{-u_n})$, a doubly stochastic Poisson process has conditional transform

$$\mathbb{E}(e^{-u \cdot (N_T - N_t)} \mid \mathcal{F}_t \vee \mathcal{G}_T) = e^{-\phi(u) \cdot (A_T - A_t)}. \tag{4.11}$$

PROPOSITION 4.3. *Suppose N is a doubly stochastic Poisson process relative to $\mathbb{G} \subset \mathbb{F}$. Also suppose the discount rate process r is $\mathbb{G}$-adapted and that $Y \in \mathcal{G}_T$. Then*

$$\mathcal{L}_t^u(v, z, w, T) = \mathcal{L}_t^{0_d}(v, z, w, T) \tag{4.12}$$

for all $u, v \in \mathbb{R}_+^d$, $t \leq T$, and $z, w \geq 0$.

Proof. By the definition of the measure $\mathbb{P}^u$ and a doubly stochastic Poisson process, iterated expectations, and the measurability properties of (A, r, Y), we have

$$\begin{aligned}\mathcal{L}_t^u(v, z, w, T) &= \mathbb{E}\left[\frac{Z_T(u)}{Z_t(u)} \exp\left(-w\int_t^T r_s ds\right) e^{-zY - v\cdot(A_T - A_t)} \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\exp\left(-w\int_t^T r_s ds\right) e^{-zY - (v-\psi(u))\cdot(A_T - A_t)} \mathbb{E}(e^{-u\cdot(L_T - L_t)} \mid \mathcal{F}_t \vee \mathcal{G}_T) \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\exp\left(-w\int_t^T r_s ds\right) e^{-zY - v\cdot(A_T - A_t)} \,\middle|\, \mathcal{F}_t\right],\end{aligned}$$

where the last equality is due to the independence of the loss magnitudes ℓ_t^n across i and n, and the doubly stochastic property (4.11) as well as the fact that conditional on $\mathcal{F}_t \vee \mathcal{G}_T$, the components of $N_T - N_t$ are mutually independent:

$$\begin{aligned}\mathbb{E}(e^{-u\cdot(L_T - L_t)} \mid \mathcal{F}_t \vee \mathcal{G}_T) &= \textstyle\prod_{n=1}^d \mathbb{E}\big(e^{\log(1-\psi_n(u_n))(N_T^n - N_t^n)} \mid \mathcal{F}_t \vee \mathcal{G}_T\big) \\ &= \prod_{n=1}^d e^{-\psi_n(-\log(1-\psi_n(u_n)))(A_T^n - A_t^n)} \\ &= e^{-\psi(u)\cdot(A_T - A_t)}.\end{aligned}$$

This completes the proof. □

The key property of a doubly stochastic Poisson process N is that its compensator A is adapted to a filtration $\mathbb{G} \subset \mathbb{F}$ with the property that an event $\{N_t^n = m\}$ is never contained in $\mathcal{G}_t$. Therefore, a doubly stochastic Poisson process cannot be self- or cross-exciting, i.e., N cannot influence A. In the general case, the arrivals of N may have an impact on the dynamics of A, as in the default timing models developed by Arnsdorf and Halperin (2008), Duffie et al. (2009), Errais, Giesecke, and Goldberg (2010), Jarrow and Yu (2001) and many others.[1] Then, the measure change is necessary. The measure change is also required in case N is doubly stochastic relative to $\mathbb{G}$ but the discount rate is not $\mathbb{G}$-adapted or the payoff Y is not $\mathcal{G}_T$-measurable. This occurs in a situation with flight-to-quality effects, where the risk-free rate responds to defaults, or in a situation in which the payoff Y depends on L_T or N_T. Section 5 provides an example.

It turns out that the point process transform can always be expressed in terms of the Laplace transform $\mathcal{L}_t^{0_d}(v, z, w, T)$ of the compensator under the reference measure $\mathbb{P}$, even if N is not doubly stochastic. However, the expression is complicated and difficult to evaluate, since it involves the conditional expectation of the discounted jumps common to the $\mathbb{P}$-Laplace transform and the point process.

PROPOSITION 4.4. *Letting* $Y_t = Y_t(u, z, w, T) = \mathcal{L}_t^{0_d}(\psi(u), z, w, T)$ *for* $t \le T, u \in \mathbb{R}_+^d$ *and* $z, w \ge 0$, *the point process transform satisfies*

$$\varphi_t(u, z, w, T) = Y_t + \sum_{t < s \le T} \mathbb{E}\left[\exp\left(-w\int_t^s r_v dv\right) \Delta Y_s \Delta e^{-u\cdot(L_s - L_t)} \,\middle|\, \mathcal{F}_t\right]. \tag{4.13}$$

[1] Note that the doubly stochastic property is generally not invariant under an equivalent change of measure. This is relevant in empirical applications, when L is first specified under the physical measure. Even if L is doubly stochastic under the physical measure, the change of measure to a risk-neutral pricing measure may introduce self- or cross-excitation into L, as in Kusuoka (1999).

Proof. By definition of $\mathcal{L}_t^{0_d}(v, z, w, T)$, for fixed (u, z, w, T) we have that

$$Y_t = \mathbb{E}\left[\exp\left(-w\int_t^T r_s ds - zY\right) e^{-\psi(u)\cdot(A_T - A_t)} \,\middle|\, \mathcal{F}_t\right]$$

and it is obvious that the process $(V_t)_{t\le T}$ defined by

$$\begin{aligned} V_t &= Y_t \exp\left(-w\int_0^t r_s ds - \psi(u)\cdot A_t\right) \\ &= \mathbb{E}\left[\exp\left(-w\int_0^T r_s ds - zY\right) e^{-\psi(u)\cdot A_T} \,\middle|\, \mathcal{F}_t\right] \end{aligned}$$

is a martingale. If moreover we define

$$U_t = Y_t \exp\left(-w\int_0^t r_s ds - u\cdot L_t\right)$$

then by comparison of U_t and V_t we see that

$$U = VZ(u)$$

where $Z(u)$ is the Radon–Nikodym density process (3.1). By integration by parts,

$$U_t = Y_0 + \int_0^t V_{s-} dZ_s(u) + \int_0^t Z_{s-}(u) dV_s + \sum_{0<s\le t} \Delta V_s \Delta Z_s(u)$$

where

$$\begin{aligned} \Delta V_s &= \Delta Y_s \exp\left(-w\int_0^s r_v dv - \psi(u)\cdot A_s\right) \\ \Delta Z_s(u) &= (\Delta e^{-u\cdot L_s}) \exp(\psi(u)\cdot A_s). \end{aligned}$$

Since both V_t and $Z_t(u)$ are martingales for $t \le T$, we have

$$\begin{aligned} \mathbb{E}(U_T|\mathcal{F}_t) &= U_t + \mathbb{E}\left(\sum_{t<s\le T} \Delta V_s \Delta Z_s(u) | \mathcal{F}_t\right) \\ &= U_t + \sum_{t<s\le T} \mathbb{E}\left[\Delta Y_s \Delta(e^{-u\cdot L_s}) \exp\left(-w\int_0^s r_v dv\right) \,\middle|\, \mathcal{F}_t\right]. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathbb{E}(U_T|\mathcal{F}_t) &= \mathbb{E}\left[Y_T \exp\left(-w\int_0^T r_s ds - u\cdot L_T\right) \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\exp\left(-w\int_t^T r_s ds - zY\right) e^{-u\cdot(L_T - L_t)} \,\middle|\, \mathcal{F}_t\right] \exp\left(-w\int_0^t r_s ds - u\cdot L_t\right) \\ &= \varphi_t(u, z, w, T) \exp\left(-w\int_0^t r_s ds - u\cdot L_t\right). \end{aligned}$$

Therefore,

$$\varphi_t(u, z, w, T) = Y_t + \sum_{t<s\le T} \mathbb{E}\left[\exp\left(-w\int_t^s r_v dv\right) \Delta Y_s \Delta e^{-u\cdot(L_s - L_t)} \,\middle|\, \mathcal{F}_t\right]$$

as claimed. □

By Proposition 4.3, the conditional expectation of the common jumps in (4.13) vanishes if the counting process N is doubly stochastic relative to a subfiltration $\mathbb{G}$, the discount rate r is $\mathbb{G}$-adapted, and the payoff $Y \in \mathcal{G}_T$. If one of these conditions is violated, then the conditional expectation is nontrivial, and it is not obvious how to evaluate it. Our equivalent measure change eliminates the need to compute this conditional expectation, and leads to an expression for the point process transform that is convenient to evaluate even in the case that the point process is not doubly stochastic.

Proposition 4.4 is related to Proposition 1 of Duffie, Schroder, and Skiadas (1996), which expresses the price of a security exposed to the default of an issuer in terms of a discounted $\mathbb{P}$-Laplace transform of the compensator of the issuer's default indicator process and the expected discounted jump at default of the Laplace transform. Proposition 4.4 generates this expression when taking L to be one-dimensional and letting $u \to \infty$.

5. AN EXAMPLE

To illustrate the properties of the measure change and the calculation of the Laplace transform $\mathcal{L}_t^u(v, z, w, T)$, we consider a basic one-dimensional formulation. Let X be a risk factor solving the SDE

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t + \delta dL_t, \quad X_0 \in \mathbb{R}, \tag{5.1}$$

where W is a standard Brownian motion in $\mathbb{R}$, the jump sensitivity $\delta \geq 0$, the compensator $A = \int_0^{\cdot} \Lambda(X_s)ds$ for a suitable function $\Lambda : \mathbb{R} \to \mathbb{R}_+$, and μ and σ are suitable functions from $\mathbb{R}$ to $\mathbb{R}$. The functions (μ, σ, Λ) are chosen such that A_T satisfies (2.1) for $T > 0$. Special cases of L include the compound Poisson process ($\sigma(x) = \delta = 0$), the compound birth process ($\mu(x) = \sigma(x) = 0$), and the compound Hawkes process ($\sigma(x) = 0$ and $\mu(x) = \mu_0 + \mu_1 x$ for constants μ_0 and μ_1). Because L has an influence on X, and X governs the intensity $\lambda = \Lambda(X)$, the point process L is self-exciting. The jump sizes of L, which represent the loss magnitudes, govern the response of λ to arrivals of N. This feature generates a negative dependence between recovery and default rates, consistent with the empirical observations of Altman et al. (2005). Our specification is completed by taking the discount rate $r = R(X)$ and payoff $Y = y(X)$ for functions R and y from $\mathbb{R}$ to $\mathbb{R}_+$. This formulation implies that also the risk-free rate responds to defaults, allowing us to model flight-to-quality effects.

The $\mathbb{P}^u$-Laplace transform (4.3) of A takes the form

$$\mathcal{L}_t^u(v, z, w, T) = \mathbb{E}^u\left[\exp\left(-\int_t^T [wR + v\Lambda](X_s)ds\right) e^{-zy(X_T)} \,\middle|\, \mathcal{F}_t\right]. \tag{5.2}$$

To compute this transform, we note that Proposition 4.2 implies that the $\mathbb{P}^u$-dynamics of the state X are described by the SDE (5.1), where W is a $\mathbb{P}^u$-Brownian motion, and L is a point process whose jumps arrive with $\mathbb{P}^u$-intensity $(1 - \psi(u))\Lambda(X)$ and whose jump magnitudes have $\mathbb{P}^u$-Laplace transform (4.9). The measure change influences only the jump frequency and jump amplitude of X. If $\delta = 0$, X does not jump and N is a $\mathbb{P}$-doubly stochastic Poisson process relative to the filtration $\mathbb{G}$ generated by W. In this case, the dynamics of X are unaffected by the measure change, and (5.2) agrees with its counterpart $\mathcal{L}_t^0(v, z, w, T)$ under $\mathbb{P}$, for any $u \geq 0$.

To illustrate the impact of the measure change on the calculation of (5.2) in a familiar case, consider the affine specification

$$\mu(x) = K_0 + K_1 x, \quad \sigma^2(x) = H_0 + H_1 x, \quad \Lambda(x) = \Lambda_0 + \Lambda_1 x,$$
$$R(x) = R_0 + R_1 x, \quad y(x) = y_0 + y_1 x.$$

Applying Proposition 1 of Duffie, Pan, and Singleton (2000) with respect to the equivalent probability measure $\mathbb{P}^u$, we obtain $\mathcal{L}_t^u(v, z, w, T) = \exp(\alpha(t) + \beta(t)X_t)$, where the functions $\beta(t) = \beta(u, v, z, w, t, T)$ and $\alpha(t) = \alpha(u, v, z, w, t, T)$ satisfy the ODEs

$$\partial_t \beta(t) = w R_1 + v\Lambda_1 - K_1\beta(t) - \frac{1}{2}H_1\beta(t)^2 + \Lambda_1(1 - \psi(u))\psi^u(\delta\beta(t))$$

$$\partial_t \alpha(t) = w R_0 + v\Lambda_0 - K_0\beta(t) - \frac{1}{2}H_0\beta(t)^2 + \Lambda_0(1 - \psi(u))\psi^u(\delta\beta(t))$$

with boundary conditions $\beta(T) = -zy_1$ and $\alpha(T) = -zy_0$, where

$$\psi^u(v) = 1 - \mathbb{E}^u\big(\exp\big(-v\ell_1^n\big)\big) = \frac{\psi(v+u) - \psi(u)}{1 - \psi(u)}$$

is the counterpart to ψ under the measure $\mathbb{P}^u$; see (4.9). By Proposition 4.1, the transform of the point process L specified by (5.1) is given by

$$\varphi_t(u, z, w, T) = \mathcal{L}_t^u(\psi(u), z, w, T) = \exp(a(t) + b(t)X_t),$$

where $b(t) = \beta(u, \psi(u), z, w, t, T)$ and $a(t) = \alpha(u, \psi(u), z, w, t, T)$ solve the ODEs

$$\partial_t b(t) = w R_1 - K_1 b(t) - \frac{1}{2}H_1 b(t)^2 + \Lambda_1\psi(u + \delta b(t))$$

$$\partial_t a(t) = w R_0 - K_0 b(t) - \frac{1}{2}H_0 b(t)^2 + \Lambda_0\psi(u + \delta b(t))$$

with boundary conditions $b(T) = -zy_1$ and $a(T) = -zy_0$. If $\delta = 0$, then the ODEs describe the $\mathbb{P}$-transform of A evaluated at $\psi(u)$, as prescribed by Propositions 4.1 and 4.3. If $u \to \infty$, then the ODEs describe the transform of A under the absolutely continuous limiting measure $\mathbb{P}^\infty$, evaluated at 1. Under $\mathbb{P}^\infty$, L does not jump prior to T, so in that case the ODEs correspond to an affine diffusion for X.

6. APPLICATIONS

We show how the point process transform $\varphi_t(u, z, w, T)$ can be used to treat various applications in single-name and portfolio credit risk. Several important valuation relations are easily obtained from $\varphi_t(u, z, w, T)$. Since $\lim_{u\to\infty}\exp(-u \cdot x) = 1_{\{x=0_d\}}$ for $x \in \mathbb{R}_+^d$, by dominated convergence we have

$$\mathbb{E}\left[\exp\left(-w\int_t^T r_s ds\right)e^{-zY}1_{\{L_T - L_t = 0_d\}}\,\middle|\,\mathcal{F}_t\right] = \lim_{u\to\infty}\varphi_t(u, z, w, T). \tag{6.1}$$

To illustrate applications of formula (6.1), let r represent the risk-free short rate, and let $\mathbb{P}$ be a pricing measure relative to r. Take $d = 1$ and consider a firm defaulting at $\tau = \inf\{t > 0 : N_t > 0\}$, the first jump time of N.

Defaultable asset with zero recovery. Consider a security paying a $\mathbb{P}$-integrable amount Y at T in case the issuer survives to T and 0 otherwise. At time $t \leq T$ and on the event $\{\tau > t\}$, the value of this security is given by

$$p_t(Y, T) = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) Y 1_{\{\tau > T\}} \,\middle|\, \mathcal{F}_t\right] = -\partial_z \lim_{u\to\infty} \varphi_t(u, z, 1, T)|_{z=0}. \tag{6.2}$$

For $r = 0$ and $Y = 1$, formula (6.2) represents the conditional probability at t that the firm survives to time T, given survival to t.

Defaultable asset with random recovery at default. Consider a security that pays an amount ρ_τ at τ if default occurs before T and zero otherwise, where ρ is a bounded and predictable process that may depend on N. Suppose N has nonnegative and bounded intensity λ, i.e., $A = \int_0^\cdot \lambda_s ds$. Using the fact that $N \wedge 1 - \int_0^\cdot \lambda_s 1_{\{\tau > s\}} ds$ is a martingale, at $t \leq T$ and on the event $\{\tau > t\}$, this security is valued at

$$\begin{aligned}
\mathbb{E}\left[\exp\left(-\int_t^\tau r_s ds\right) \rho_\tau 1_{\{\tau \leq T\}} \,\middle|\, \mathcal{F}_t\right] &= \mathbb{E}\left[\int_t^T \exp\left(-\int_t^s r_v dv\right) \rho_s d(N \wedge 1)_s \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_t^T \exp\left(-\int_t^s r_v dv\right) \rho_s \lambda_s 1_{\{\tau > s\}} ds \,\middle|\, \mathcal{F}_t\right] \\
&= \int_t^T p_t(\rho_s \lambda_s, s)\, ds. \qquad (6.3)
\end{aligned}$$

Alternative formulae can be developed in the case that an intensity λ does not exist.

Credit swap. Consider a unit notional credit swap, in which the protection seller pays ρ_τ at default if $\tau \leq T$, and in which the protection buyer pays a premium S at dates (t_m) until $\tau \wedge T$. Using formulae (6.2) and (6.3), denoting by c_m the day count fraction for period m and ignoring premium accruals, the swap rate $S_t = S$ at time t satisfies

$$S \sum_{t_m \geq t} c_m p_t(1, t_m) = \int_t^T p_t(\rho_s \lambda_s, s)\, ds.$$

Equity derivatives with jump to default. Formula (6.1) facilitates a key calculation arising in equity derivative models with jump to default, including those of Carr and Madan (2010), Carr and Linetsky (2006), Linetsky (2006), Mendoza, Carr, and Linetsky (2010), and others. In these models, the value X of a stock subject to default at $\tau = \inf\{t > 0 : N_t > 0\}$ solves $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t - X_{t-}dN_t$ where W is a standard Brownian motion and μ and σ are suitable functions. The counting process N has intensity $\lambda = \Lambda(X)$ for a suitable decreasing function Λ. At τ, X jumps to 0. The value at $t \leq T$ of a derivative that pays $g(X_T)1_{\{\tau > T\}}$ at T is given by (6.2) with $Y = g(X_T)$. Under the limiting measure $\mathbb{P}^\infty$ appearing in the calculation of the right-hand side of (6.2) based on formula (4.2), X solves $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$ on $[0, T]$, reducing the valuation problem to one involving a diffusion. This approach to the valuation of a derivative written on a defaultable stock is simpler than the approach taken in the aforementioned papers. The measure change approach eliminates the need to perform a progressive filtration enlargement and to introduce a predefault stock price process

distinct from X. It also covers the multidimensional case of d defaultable stock price processes that are correlated.

Bottom-up approach. Formula (6.1) has also applications in a multi-name setting. Here, the first jump time $\tau_n = \inf\{t > 0 : N_t^n > 0\}$ of N^n models the default time of name n in a portfolio of $d \geq 1$ names, and $C_T = \ell_1^1 1_{\{\tau_1 \leq T\}} + \cdots + \ell_1^d 1_{\{\tau_d \leq T\}}$ is the loss in the portfolio at T. This formulation allows one to address a fully heterogeneous portfolio in which each name has distinct default arrival and loss magnitude characteristics. For $t \leq T$ and $v \geq 0$, adopting the convention $\prod_{n\in\emptyset} a_n = 1$ for any sequence (a_n) we have

$$\begin{aligned}\exp(-v(C_T - C_t)) &= \prod_{n\notin S(t)} \left[\left(1 - \exp\left(-v\ell_1^n\right)\right) 1_{\{N_T^n = 0\}} + \exp\left(-v\ell_1^n\right)\right] \\ &= c_\emptyset(v) + \sum_{S_k(t)} c_{S_k(t)}(v) 1_{\{N_T^n = 0, n\in S_k(t)\}},\end{aligned}$$

where $S(t) = \{1 \leq n \leq d : N_t^n > 0\} \in \mathcal{F}_t$, the sum ranges over all subsets $S_k(t)$ of $\{1, \ldots, d\} \setminus S(t)$, and the functions

$$c_{S_k(t)}(v) = \prod_{n\in S_k(t)} \left(1 - \exp\left(-v\ell_1^n\right)\right) \prod_{n\notin S_k(t)\cup S(t)} \exp\left(-v\ell_1^n\right).$$

Therefore, if the loss magnitudes ℓ_1^n are independent of r and Y, formula (6.1) yields

$$\begin{aligned}\Phi_t(v, z, w, T) &= \mathbb{E}\left[\exp\left(-w\int_t^T r_s ds\right) e^{-zY - v(C_T - C_t)} \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}(c_\emptyset(v)|\mathcal{F}_t)\mathbb{E}\left[\exp\left(-w\int_t^T r_s ds\right) e^{-zY} \,\middle|\, \mathcal{F}_t\right] \\ &+ \sum_{S_k(t)} \mathbb{E}(c_{S_k(t)}(v) \mid \mathcal{F}_t)\mathbb{E}\left[\exp\left(-w\int_t^T r_s ds\right) e^{-zY} 1_{\{N_T^n - N_t^n = 0, n\in S_k(t)\}} \,\middle|\, \mathcal{F}_t\right] \\ &= \sum_{S_k(t)} \mathbb{E}(c_{S_k(t)}(v) \mid \mathcal{F}_t) \lim_{\substack{u_n\to\infty \\ n\in S_k(t)}} \varphi_t(u, z, w, T)|_{u_n = 0, n\notin S_k(t)}\end{aligned}$$

where $\mathbb{E}(c_{S_k(t)}(v) \mid \mathcal{F}_t)$ is a function of the $\psi_n(v)$ and the sets $S(t)$ and $S_k(t)$. To compute the conditional distribution of $C_T - C_t$ and the prices of securities referenced on the portfolio, we invert $\Phi_t(u, z, w, T)$ as indicated below.

Top-down approach. An alternative formulation of the multi-name problem takes $d = 1$ and supposes that the process L records the loss due to default in the reference portfolio. This formulation is appropriate for a relatively large and homogenous portfolio, and leads to convenient valuation relations for securities exposed to portfolio credit risk, i.e., derivatives written on L. For some derivative payoff function $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and for fixed parameters (z, w, t, T), let

$$G_h(k) = \mathbb{E}\left[\exp\left(-w\int_t^T r_s ds\right) e^{-zY} h(L_T - L_t, k) \,\middle|\, \mathcal{F}_t\right]; \tag{6.4}$$

the variable k describes some contract feature, such as a strike price. Let $\mathbb{P}$ be a pricing measure relative to the risk-free rate r. Then, for $w = 1$, $G_h(k)$ represents the value at $t \leq T$ of a security paying $e^{-zY} h(L_T - L_t, k)$ at T. Following the approach of Carr and Madan (1999) and Lee (2004), we consider the Fourier transform of the damped value

$\exp(\alpha k)G_h(k)$, where $\alpha \in \mathbb{R}$ is some constant. If α is chosen appropriately, then the Fourier transform

$$c_{\alpha,h}(v) = \int_{-\infty}^{\infty} e^{ivk} e^{\alpha k} G_h(k) dk \tag{6.5}$$

is well defined for $v \in \mathbb{R}$. The transform can then be inverted to obtain

$$G_h(k) = \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} \mathrm{Re}\big(e^{-ivk} c_{\alpha,h}(k)\big)\, dv. \tag{6.6}$$

Option on portfolio loss. Consider a European call option on L with strike $k \geq 0$ and expiration date T, for which $h(l, k) = (l - k)^+$. We have

$$\begin{aligned} c_{\alpha,h}(v) &= \mathbb{E}\left[\exp\left(-w\int_t^T r_s ds - zY\right)\int_{-\infty}^{L_T - L_t} e^{(\alpha+iv)k}(L_T - L_t - k)dk \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\exp\left(-w\int_t^T r_s ds - zY\right)\frac{e^{(\alpha+iv)(L_T-L_t)}}{(\alpha+iv)^2} \,\middle|\, \mathcal{F}_t\right] \\ &= \frac{g(-\alpha - iv)}{(\alpha+iv)^2} \end{aligned}$$

provided that $\alpha > 0$ is chosen such that $-\alpha - iv \in \mathcal{C}_L = \{c \in \mathbb{C} : \mathrm{Re}(c) \in D_L\}$, where D_L is the interior of the set $\{v \in \mathbb{R} : \mathbb{E}(\exp(-vL_T)) < \infty\}$, and $g : \mathcal{C}_L \to \mathbb{C}$ is the analytic function defined for a fixed set of parameters (z, w, t, T) by

$$g(c) = \mathbb{E}\left[\exp\left(-w\int_t^T r_s ds\right) e^{-zY - c(L_T - L_t)} \,\middle|\, \mathcal{F}_t\right].$$

Taking $g(\cdot)$ as the extension of $\varphi_t(\cdot, z, w, T)$ to $\mathcal{C}_L$,[2] the value of a call on L_T with strike K can be expressed as

$$\begin{aligned} C_t(K, T) &= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)(L_T - K)^+ \,\middle|\, \mathcal{F}_t\right] \\ &= \frac{e^{-\alpha(K-L_t)}}{\pi}\int_0^{\infty} \mathrm{Re}\left(\frac{\varphi_t(-\alpha - iv, 0, 1, T)e^{-iv(K-L_t)}}{(\alpha+iv)^2}\right) dv. \end{aligned} \tag{6.7}$$

The price of a put option on L_T can be found by put-call parity. An option is a basic building block of many more complex portfolio credit derivatives such as credit indexes, tranches, or options on these instruments.

Tranche. A tranche swap on a portfolio with unit notional is specified by a lower attachment point $\underline{K} \in [0, 1]$, an upper attachment point $\bar{K} \in (\underline{K}, 1]$, and a maturity date T. The tranche protection seller agrees to cover portfolio losses as they occur, given that the cumulative losses are larger than $\underline{K}$ but do not exceed $\bar{K}$. Letting L describe the portfolio loss, the cumulative payments U_t at t are

$$U_t = (L_t - \underline{K})^+ - (L_t - \bar{K})^+,$$

[2] Under mild assumptions, $\mathcal{L}_t^u(\psi(u), z, w, T)$ has a Taylor expansion for $u \geq 0$. Evaluating this expansion at $c \in \mathcal{C}_L$ yields an expression for $g(c)$. Thus, we obtain an expression for $\varphi_t(\cdot, z, w, T)$ on $\mathcal{C}_L$.

which is the payoff to a call spread on L with strike pair $(\underline{K}, \bar{K})$ and exercise date t. By integration by parts, the value at $t \leq T$ of the protection seller's payments is given by

$$D_t = \mathbb{E}\left[\int_t^T \exp\left(-\int_t^s r_u du\right) dU_s \,\middle|\, \mathcal{F}_t\right]$$

$$= C_t(\underline{K}, T) - C_t(\bar{K}, T) - U_t + \int_t^T (B_t(\underline{K}, s) - B_t(\bar{K}, s))\, ds,$$

where $C_t(K, T)$ is given by (6.7), and, mimicking the pricing approach followed above, under suitable conditions, $B_t(K, T)$ can be expressed as

$$B_t(K, T) = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) r_T (L_T - K)^+ \,\middle|\, \mathcal{F}_t\right]$$

$$= \frac{e^{-\alpha(K-L_t)}}{\pi} \int_0^\infty \mathrm{Re}\left(-\frac{\partial_z \varphi_t(-\alpha - iv, z, 1, T)|_{z=0}\, e^{-iv(K-L_t)}}{(\alpha + iv)^2}\right) dv,$$

where $\varphi_t(-\alpha - iv, z, 1, T)$ is evaluated for $Y = r_T$, and α is an appropriately chosen damping constant. The tranche protection buyer agrees to make premium payments at dates (t_m) that are proportional to the difference between the tranche notional $\bar{K} - \underline{K}$ and the tranche loss U. If S is the proportionality factor and c_m is the day count fraction for the mth coupon period, then the value at $t \leq T$ of these payments is

$$P_t(S) = \mathbb{E}\left[\sum_{t_m \geq t} \exp\left(-\int_t^{t_m} r_s ds\right) S c_m (\bar{K} - \underline{K} - U_{t_m}) \,\middle|\, \mathcal{F}_t\right]$$

$$= S \sum_{t_m \geq t} c_m ((\bar{K} - \underline{K}) \varphi_t(0, 0, 1, t_m) - C_t(\underline{K}, t_m) + C_t(\bar{K}, t_m)).$$

The fair tranche spread at t is the solution $S = S_t$ to the equation $D_t = P_t(S)$. Similar calculations lead to the fair spread on a credit index swap. Moreover, the analysis can be extended to $d > 1$. In a multidimensional setting, the components of L represent the losses in d subportfolios. The subportfolios have different default arrival and loss magnitude characteristics, allowing one to address the heterogeneity of a reference portfolio. The cash flow to a tranche is a function of the total portfolio loss $1_d \cdot L$.

Option on portfolio loss with counterparty risk. The point process transform can be used to treat valuation problems that involve counterparty risk. Here, a counterparty to a credit derivative may default on its obligation. A counterparty's risk of default may be correlated with the credit risk associated with the reference portfolio. In particular, a counterparty's conditional default probability may jump at a default in the reference portfolio, as in Jarrow and Yu (2001). To illustrate, suppose the issuer of an option on portfolio loss is subject to default risk. For $L = (L^1, L^2)$, let L^1 be the portfolio loss and let $\tau_2 = \inf\{t > 0 : L_t^2 > 0\}$ be the default time of the option writer. Then the option pays $(L_T^1 - K)^+$ at T if $\tau_2 > T$ and 0 otherwise. To value this option, let

$$\overline{\varphi}_t(u_1, T) = \lim_{u_2 \to \infty} \varphi_t((u_1, u_2), 0, 1, T)$$

$$= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) e^{-u_1(L_T^1 - L_t^1)} 1_{\{L_T^2 - L_t^2 = 0\}} \,\middle|\, \mathcal{F}_t\right]$$

for $t \leq T$. On the set $\{\tau_2 > t\}$, and under suitable conditions and an appropriately chosen damping constant α, the price of the option at $t \leq T$ can be expressed as

$$\mathbb{E}\left[\exp\left(-\int_t^T r_s ds \right) (L_T^1 - K)^+ 1_{\{\tau_2 > T\}} \,\middle|\, \mathcal{F}_t \right]$$
$$= \frac{e^{-\alpha(K-L_t)}}{\pi} \int_0^\infty \mathrm{Re}\left(\frac{\overline{\varphi}_t(-\alpha - iv, T) e^{-iv(K-L_t)}}{(\alpha + iv)^2} \right) dv.$$

Following the reasoning leading to formula (6.3), we can extend to the case that the option holder recovers some amount at τ_2. Moreover, we can extend the tranche and credit swap valuation formulae developed above to incorporate the effects of default risk of both the seller and the buyer of protection (bilateral counterparty risk).

7. CONCLUSION

This paper develops a formula for a transform of a vector point process with totally inaccessible event times. An equivalent change of probability measure reduces the calculation of the point process transform to the calculation of a Laplace transform of the point process compensator. This Laplace transform is analogous to the value of a security providing a single cash flow at a future horizon. The calculation of this value is standard for a wide range of model specifications. Our transform formula extends this computational tractability to a point process and its applications.

We show how to use the point process transform to address a number of applications in single-name and portfolio credit risk, including the valuation of credit derivatives such as credit and tranche swaps, and the pricing of equity derivatives on defaultable stocks. Our transform formula enables us to analyze a setting that takes account of many empirically relevant features. For example, default, recovery and risk-free interest rates may be correlated and depend on past defaults, allowing for flight-to-quality and self- and cross-exciting effects. Moreover, a counterparty to a derivative contract may be subject to default risk.

Our results have potential applications in several other areas, including the stochastic modeling and statistical analysis of limit order books. Bowsher (2007), Engle (2000) and Engle and Russell (1998) pioneered an econometric approach in which order arrivals form a point process whose arrival intensity is estimated from order timing data using likelihood methods. Our results allow one to widen the scope of this approach to multivariate, self-, and cross-exciting arrival intensity models with more realistic features while maintaining computational tractability of estimation, testing, and prediction.

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