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Time-Changed Birth Processes and Multiname Credit Derivatives

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A credit investor such as a bank granting loans to firms or an asset manager buying corporate bonds is exposed to correlated corporate default risk. A multiname credit derivative is a financial security that allows the investor to transfer this risk to the credit market. In this paper, we study the valuation and risk analysis of multiname derivatives. To capture the complex economic phenomena that drive the pricing of these securities, we introduce a time-changed birth process as a probabilistic model of correlated event timing. The self-exciting property of a time-changed birth process captures the feedback from events that is often observed in credit markets. The stochastic variation of arrival rates between events captures the exposure of firms to common economic risk factors. We derive a closed-form expression for the distribution of a time-changed birth process, and develop analytically tractable pricing relations for a range of multiname derivatives valuation problems. We illustrate our results by calibrating a tranche forward and option pricer to market rates of index and tranche swaps.

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1. Introduction

A credit investor such as a bank granting loans to firms or an asset manager buying corporate bonds is exposed to correlated corporate default risk. A multiname credit derivative is a financial security that allows the investor to transfer this risk to another investor who is willing to bear it. Through such a derivative, an investor can buy or sell default insurance on a portfolio of credit-sensitive instruments. Industry sources estimate the size of the multiname derivatives market at the end of 2006 to be nearly $8 trillion, representing more than a 35% increase over the prior year.

The price for multiname default insurance is a function of the portfolio constituent default risks and their dependence structure. Issuers of credit-sensitive instruments such as bonds or loans are exposed to common or correlated economic risk factors such as interest rates or commodity prices. The random movement of these risk factors generates correlated changes in firms' conditional default probabilities, and the cyclical pattern in the time-series behavior of aggregate default rates. Due to the complex web of business, legal, and informational relationships in the economy, defaults have a direct impact on the prospects of the surviving firms. For example, the collapse of automotive manufacturer Delphi in 2005 severely affected General Motors (GM), whose production critically depended on Delphi's timely supply of parts. In response to the event, investors immediately demanded a higher default insurance premium for GMs, reflecting the sudden increase in GM's likelihood to fail. Collin-Dufresne et al. (2003) and Jorion and Zhang (2007) show that these feedback effects are not anecdotal.

The valuation and risk analysis of a multiname derivative is challenging in light of the complex economic phenomena that drive correlated corporate default risk. We require a probabilistic model of event timing that captures these phenomena and leads to computationally tractable pricing and hedging relations. In this paper, we introduce a time-changed birth process to model correlated event arrivals. A time-changed birth process is obtained by evaluating a classical birth process at a random clock. The time change generates a point process with attributes that are relevant for many applications, including portfolio credit risk. Like a classical birth process, a time-changed birth process is self-exciting: an arrival has an impact on the intensity, or conditional arrival rate of events. Therefore, an event increases the likelihood of further arrivals. Unlike a classical birth process, a time-changed birth process has an intensity that is allowed to move randomly between events. This stochastic variation reflects the sensitivity of arrival rates to exogenous risk factors that fluctuate randomly.

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The time change preserves the analytical tractability of the classical birth process. We derive a closed-form expression for the distribution of a time-changed birth process in terms of the Laplace transform of the time change. The Laplace transform is a familiar expression in finance. It is analogous to the price of a security that pays one dollar at a future time in an economy where the risk-free rate of interest is proportional to the rate of the time change. The explicit calculation of this price is well understood for a wide range of interest rate model specifications. This observation suggests a wealth of candidate models for the rate of the time change for which the Laplace transform, and hence the distribution of the time-changed birth process, can be explicitly calculated. Examples include affine and quadratic models, for which the Laplace transform is an exponentially affine or quadratic function of the state.

The computational benefits extend to our applications in portfolio credit risk, in which a compound time-changed birth process describes the financial loss due to default in a portfolio of loans or bonds. We derive a closed-form expression for the price of a call option on the portfolio loss in terms of the Laplace transform of the time change and the characteristic function of the random loss at an event. The characteristic function can be calculated explicitly for many parametric recovery distributions that are of economic interest. The option formula is the key to valuing a multiname derivative, which is a security whose payoff can be expressed in terms of linear combinations of options on the portfolio loss. Example contracts that we analyze include index and tranche swaps, which are the most important and most liquid standardized multiname derivatives. These swaps are bilateral financial contracts that are referenced on portfolios of North American and European issuers, in which one party buys insurance from the other party against the default losses in the portfolio. Forwards and options on these swaps are examples of more exotic multiname derivatives that we analyze. These contracts allow an investor to express a view on the volatility of portfolio loss. The static copula models of correlated default that are widely used in the financial industry are inappropriate to analyze options on index and tranche swaps.

To illustrate model specification and security valuation, we calibrate a time-changed birth process model to market index and tranche swap rates. The rate of the time change is assumed to follow a mean-reverting Feller diffusion, a standard model in finance for interest rates. This choice guarantees analytical valuation formulae for index and tranche swaps. We obtain accurate fits to market index and tranche swap rates of several maturities. The calibrated model is then used to price tranche forwards and options. The results indicate the significance of event feedback and stochastic variation of the interevent intensity for the valuation of exotic multiname derivatives.

A time-changed birth process can alternatively be termed a *doubly stochastic birth process* because conditional on the time change, its arrivals form a time-inhomogeneous birth process. There is event feedback even if the uncertainty about the risk factors driving the time change is removed. This property distinguishes the doubly stochastic birth process from its narrower but widely used descendant, the doubly stochastic Poisson or Cox process. The Cox process arises in the special case where the original birth process is in fact a simple Poisson process. Here, conditional on the risk factors, events form a time-inhomogeneous Poisson process. The interarrival times are conditionally independent, which rules out feedback from events. Not surprisingly, the Cox process provides a poor fit to empirical data. For example, Das et al. (2007) provide evidence of historical default clustering in excess of that implied by a compatibly estimated Cox model. A self-exciting time-changed birth process provides an alternative model that allows for a more realistic degree of clustering while preserving computational tractability.

In related work, Davis and Lo (2001) model defaults by a piecewise deterministic Markov process that is self-exciting. Ordinary differential equations govern the default distribution. In Giesecke and Tomecek (2005), events follow a time-changed standard Poisson process, which is self-exciting if the time change depends on the Poisson process. The transform of the process is obtained by a complex-valued measure change as in Carr and Wu (2004). In this paper, we generate a self-exciting model by time-changing a birth process rather than a Poisson process. Because a birth process is already self-exciting, the time change can be chosen independently and the distribution of the loss process can be derived by elementary means. In Errais et al. (2006), the portfolio loss follows a self-exciting affine point process. Its transform is an exponentially affine function of the risk factors, with coefficients that satisfy ordinary differential equations. The price of a credit derivative incorporates the dependence structure among default and recovery rates, and is given in terms of an inverse transform. Tavella and Krekel (2006) provide a formulation in which the derivative price is governed by a system of backward partial differential equations. Brigo et al. (2006) propose alternative doubly stochastic Poisson model specifications of the portfolio loss. Longstaff and Rajan (2008) develop a three-factor model of portfolio loss in which the factors represent firm-specific, sectoral, and economywide default risks.

Although we emphasize credit derivatives in this paper, a time-changed birth process also has important potential applications in the risk management and capital provisioning of credit portfolios. Das et al. (2007) show that a doubly stochastic Poisson model similar to those widely used for the calculation of risk capital tends to underestimate the probability of large losses, and thus may lead to undercapitalization. Because it implies loss distributions with fatter tails, a self-exciting model may be more appropriate. Giesecke and Weber (2004) provide an explicit approximation to the loss distribution in an interacting Markov process model in which the intensity of a firm depends on

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the status of other firms. Duffie et al. (2006) estimate a frailty model in which individual firm intensities depend on common unobservable default covariates and are updated at events. The class of time-changed birth processes supports other self-exciting model specifications that facilitate an analytical treatment of risk measurement and capital allocation problems.

The remainder of this paper is organized as follows. Section 2 introduces a time-changed birth process and derives its basic properties. Section 3 discusses the application of a time-changed birth process to the valuation and risk analysis of multiname credit derivatives. It explains the valuation of index and tranche swaps, forwards, and options. Section 4 specifies parametric models for the time change and the loss at default that lead to analytically tractable valuation formulae. Section 5 explains the implementation and market calibration of an index and tranche swap and option pricer. Section 6 concludes.

2. Time-Changed Birth Process

This section introduces a time-changed birth process, studies its properties, and derives a formula for its conditional distribution. The classical birth process is reviewed first.

2.1. Birth Process

Consider a counting process N^0 defined on a probability space $(\Omega, \mathcal{G}, P)$ endowed with a right-continuous and complete filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$. Suppose that N^0 is a birth process with intensity, or conditional arrival rate λ^0 given by

$$\lambda_t^0 = c + \delta N_t^0,$$

where $c > 0$ and $\delta > 0$. It follows that the first event arrives with constant intensity c. At the first event, the intensity increases to $c+\delta$ and stays constant until the second event, at which time it increases to $c + 2\delta$, and so on. Thus, the birth process is *self-exciting*: an arrival increases the likelihood of further arrivals. For $\delta \to 0$, the self-exciting property vanishes and N^0 is a Poisson process with intensity c. The conditional characteristic function of the birth process takes the form

$$E[e^{iv(N_s^0 - N_t^0)} \mid \mathcal{F}_t] = (e^{iv} + (1 - e^{iv})e^{\delta(s-t)})^{-C_t^0}, \tag{1}$$

where $t \leqslant s$, $v \in \mathbb{R}$, and $C_t^0 = N_t^0 + c/\delta$. The conditional distribution of N^0 is negative binomial; see Feller (1957, Chapter 17.3).[1] For $t \leqslant s$ and $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$, we have

$$P[N_s^0 - N_t^0 = k \mid \mathcal{F}_t] = \frac{\Gamma(C_t^0 + k)}{\Gamma(C_t^0)k!}(\Delta_{t,s})^{C_t^0}(1 - \Delta_{t,s})^k, \tag{2}$$

where Γ is the gamma function and $\Delta_{t,s} = \exp(-\delta(s-t))$. Thus, the birth process is a homogeneous Markov process with conditional mean and variance given by

$$E[N_s^0 - N_t^0 \mid \mathcal{F}_t] = C_t^0(e^{\delta(s-t)} - 1),$$
$$\mathrm{Var}[N_s^0 - N_t^0 \mid \mathcal{F}_t] = C_t^0(e^{\delta(s-t)} - 1)e^{\delta(s-t)}.$$

Due to the self-exciting property, the variance is always larger than the mean, and so the negative binomial distribution is overdispersed relative to the Poisson distribution. The mean and variance grow exponentially in s at rates δ and 2δ, respectively.[2]

2.2. Time Change

We propose to time change the birth process to generate a self-exciting point process whose arrival rate is allowed to move randomly between events. Let ν be a strictly positive stochastic process that is independent of the birth process N^0. We require that for each t, the variable ν_t admits a density and the variable

$$T_t = \int_0^t \nu_s \, ds \tag{3}$$

is a stopping time in the filtration $\mathbb{F}$ that is finite almost surely. In addition, we assume that (T_t) increases to infinity almost surely. Under these conditions, the family (T_t) defines a *time change*. The evolution of the *activity rate* ν controls the passage of new time T_t relative to calendar time t. We evaluate the birth process in the new time scale to obtain the *time-changed birth process* N given by

$$N_t = N_{T_t}^0.$$

The information set in the new time scale is represented by the right-continuous and complete filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geqslant 0}$ generated by the stopping time sigma-fields $\mathcal{F}_{T_t}$. We are interested in the intensity λ of the time-changed birth process relative to $\mathbb{G}$, which represents the conditional arrival rate in the new time scale. Because the original birth process is a time-changed Poisson process in the sense of Giesecke and Tomecek (2005), so is N. The time change transforming a standard Poisson process into N is given by the composition $A_t = \int_0^{T_t} \lambda_s^0 \, ds$.[3] The results in Giesecke and Tomecek (2005) imply that A is also the compensator of N in the filtration $\mathbb{G}$.[4] The time derivative of A is the intensity λ of N relative to $\mathbb{G}$, and is given by

$$\lambda_t = \nu_t \lambda_{T_t}^0 = \nu_t(c + \delta N_t). \tag{4}$$

The time-changed birth process N inherits the self-exciting property from the birth process. Its intensity is an affine function of N itself, with coefficients following independent stochastic processes driven by the activity rate. The intensity has dynamics

$$d\lambda_t = (c + \delta N_{t-})d\nu_t + \nu_{t-}\delta dN_t. \tag{5}$$

The activity rate modulates the intensity of N. For example, if ν is mean reverting, then so is λ. The mean-reversion speed and level are affine functions of N, and hence are updated at events. Similarly, λ inherits any Brownian and jump terms from ν, with coefficients that have affine dependence on N. Thus, by specifying different stochastic processes for the activity rate, we can generate a vast array of

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self-exciting point processes N that descend from the birth process. Parametric examples are considered in §4.

Conditional on a realization of the activity rate, N is a time-inhomogeneous birth process with an intensity that is deterministic between arrivals. This suggests the alternative terminology *doubly stochastic birth process* for N, in analogy to the well-known doubly stochastic Poisson process. A doubly stochastic Poisson process, or Cox process, arises in the special case where $\delta \to 0$. Then, the birth process N^0 is a Poisson process, and, conditional on a realization of ν, N is a time-inhomogeneous Poisson process.

2.3. Transform and Distribution

Time-changed birth processes are analytically tractable. Because the time change is independent of the birth process, the conditional characteristic function of the time-changed birth process N can be obtained from formula (1) by iterated expectations:

$$E[e^{iv(N_s-N_t)} \mid \mathcal{G}_t] = E[(e^{iv}+(1-e^{iv})e^{\delta(T_s-T_t)})^{-C_t} \mid \mathcal{G}_t] \quad (6)$$

for $t \leqslant s$ and $v \in \mathbb{R}$, where C is the process defined by

$$C_t = N_t + \frac{c}{\delta}.$$

Compare formulae (1) and (6). The corresponding conditional distribution of $N_s - N_t$ is completely determined by the conditional distribution of the time change, which is encoded in the Laplace transform

$$\mathcal{L}_{t,s}(u) = E[e^{-u(T_s-T_t)} \mid \mathcal{G}_t] \quad (7)$$

for $t \leqslant s$ and $u \geqslant 0$. This Laplace transform is a familiar expression in finance. It is analogous to the price at time t of a security that pays one dollar at time s in an economy where the short-term interest rate follows the process $u\nu$. The calculation of this price is well understood for a wide range of parametric models of ν; §4 gives examples.

Proposition 2.1. *For $t \leqslant s$ and $k \in \mathbb{N}$, we have the formula*

$$P[N_s - N_t = k \mid \mathcal{G}_t] = \frac{\Gamma(C_t+k)}{\Gamma(C_t)k!}\sum_{m=0}^{k}\binom{k}{m}(-1)^m \mathcal{L}_{t,s}(\delta(m+C_t)).$$

This result is a corollary of Proposition 2.2 below. It extends the analytical tractability offered by extant security pricing model specifications to a time-changed birth process N. Note that in the special case where $\delta \to 0$ and N is a doubly stochastic Poisson process, Proposition 2.1 reduces to the formula

$$P[N_s - N_t = k \mid \mathcal{G}_t] = \frac{(-1)^k}{k!}\frac{\partial^k}{\partial u^k}\mathcal{L}_{t,s}(uc)\bigg|_{u=1}.$$

If $\mathcal{L}_{t,s}(u)$ exists for $u=-\delta$ and $u=-2\delta$, then the conditional mean and variance of N are well defined and given by the following expressions:

$$E[N_s - N_t \mid \mathcal{G}_t] = C_t(\mathcal{L}_{t,s}(-\delta)-1),$$
$$\mathrm{Var}[N_s - N_t \mid \mathcal{G}_t] = C_t(\mathcal{L}_{t,s}(-2\delta)-\mathcal{L}_{t,s}(-\delta)).$$

Proposition 2.1 can be generalized to obtain the conditional joint distribution of N and ν. We express this distribution in terms of the "extended Laplace transform"

$$\mathcal{T}_{t,s}(u,v) = E[e^{-u(T_s-T_t)+iv\nu_s} \mid \mathcal{G}_t] \quad (8)$$

for $t \leqslant s$, $u \geqslant 0$, and $v \in \mathbb{R}$. The extended transform reduces to the ordinary transform (7) when $v=0$. It is analogous to the price at time t of a security that pays $\exp(iv\nu_s)$ dollars at time s in an economy where the short-term interest rate follows the process $u\nu$. This price can be calculated explicitly for a wide range of parametric models for ν.

Proposition 2.2. *For $t \leqslant s$, $u \geqslant 0$, $v \in \mathbb{R}$, and $k \in \mathbb{N}$, we have*

$$E[e^{-u(T_s-T_t)+iv\nu_s}1_{\{N_s-N_t=k\}} \mid \mathcal{G}_t] = \frac{\Gamma(C_t+k)}{\Gamma(C_t)k!}\sum_{m=0}^{k}\binom{k}{m}(-1)^m \mathcal{T}_{t,s}(u+\delta(m+C_t),v).$$

The proof is in the appendix. Proposition 2.1 above is the special case where $u=0$ and $v=0$. The joint distribution of the time-changed birth process and the activity rate is obtained by setting $u=0$, extending $\mathcal{T}_{t,s}(\delta(m+C_t),\cdot)$ to the complex plane, and applying Fourier inversion. For $t \leqslant s$, $k \in \mathbb{N}$, and $y \geqslant 0$, we get the formula

$$P[N_s - N_t = k, \nu_s \in dy \mid \mathcal{G}_t] = \frac{\Gamma(C_t+k)}{\Gamma(C_t)k!}\sum_{m=0}^{k}\binom{k}{m}\frac{(-1)^m}{2\pi}\int_{\gamma i-\infty}^{\gamma i+\infty}\mathcal{T}_{t,s}(\delta(m+C_t),z)e^{-izy}\,dz\,dy, \quad (9)$$

where γ is chosen such that $\gamma i \in S_{\mathcal{T}} \cap \{z \in \mathbb{C}\colon \mathrm{Im}(z) < 0\}$, where $S_{\mathcal{T}}$ is the region of convergence of $\mathcal{T}_{t,s}(\delta(m+C_t),z)$. A similar argument yields the joint law of the time-changed birth process, the time change and the activity rate.

3. Multiname Credit Derivatives

We propose a time-changed birth process N as a model of correlated default timing in a portfolio of credit-sensitive securities such as loans and bonds. The self-exciting property of a time-changed birth process captures the feedback of an event on the default arrival rate. Due to the complex web of business, legal, and informational relationships in the economy, a default often has a direct impact on the prospects of the surviving firms. The time change induces stochastic variation in default rates between events. This variation reflects the exposure of firms to common

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or correlated economic risk factors such as interest rates or commodity prices. The random movement of these risk factors generates correlated changes in firms' conditional default probabilities, and the cyclical pattern in the time-series behavior of aggregate default rates.

We consider the arbitrage-free valuation of multiname credit derivatives, which are also known as portfolio derivatives. These are securities whose payoffs depend on the financial loss due to default in the portfolio. Multiname credit derivatives allow investors to trade the correlated default risk in the portfolio. Index and tranche swaps are the most important and most liquid standardized portfolio derivatives. They are referenced on portfolios of North American and European issuers. Forwards and options on these contracts are examples of more exotic portfolio derivatives.

Under the absence of arbitrage and market frictions, and under mild technical conditions, there exists a risk-neutral probability measure, as shown by Harrison and Kreps (1979) and Harrison and Pliska (1981). We take the reference probability P to be such a risk-neutral measure, and we model correlated default arrivals as a time-changed birth process N on $(\Omega, \mathcal{G}, \mathbb{G}, P)$, directly under risk-neutral probabilities. Then, letting $r > 0$ be a constant risk-free interest rate, an arbitrage-free price at time t of a security paying a random amount π at some time T is given by the conditional expectation of the discounted payoff, $E[e^{-r(T-t)}\pi \mid \mathcal{G}_t]$.

3.1. Portfolio Loss Process

Consider a portfolio of credit-sensitive securities that are issued by n firms. The *default process* N^n counts the defaults in the portfolio, so N_t^n is the number of firms that have defaulted by t. We model the default process $N^n = N \wedge n$ as a time-changed birth process N stopped at the nth event. The intensity λ^n of N^n is equal to the intensity λ of N up to the nth event, at which time it falls to zero: $\lambda^n = \lambda 1_{\{N<n\}}$. The conditional distribution of N^n can be calculated from the conditional distribution of N given in Proposition 2.1. We simply adjust the probability of the final bin:

$$P_{t,s}(k) = P[N_s^n - N_t^n = k \mid \mathcal{G}_t] = \begin{cases} P[N_s - N_t = k \mid \mathcal{G}_t] & \text{if } k < n - N_t^n, \\ P[N_s - N_t \geqslant k \mid \mathcal{G}_t] & \text{if } k = n - N_t^n, \\ 0 & \text{if } k > n - N_t^n. \end{cases} \tag{10}$$

The *loss process* L^n records financial loss: the random variable L_t^n is the cumulative portfolio loss due to default at time t. Letting l_k denote the random loss at the kth default, L^n is a compound time-changed birth process stopped at the nth event:

$$L^n = \sum_{k=1}^{N^n} l_k.$$

We assume that the l_k are mutually independent and independent of the birth process and the time change. They have common characteristic function $g(v) = E[e^{ivl_1}]$, where $v \in \mathbb{R}$. Then, the conditional characteristic function of the loss process can be expressed in terms of the conditional distribution of the default process. By conditioning,

$$E[e^{iv(L_s^n - L_t^n)} \mid \mathcal{G}_t] = \sum_{k=0}^{n-N_t^n} P_{t,s}(k)(g(v))^k \tag{11}$$

for $t \leqslant s$ and $v \in \mathbb{R}$. The conditional expected future loss is obtained by differentiation, and the conditional loss distribution can be recovered by inversion.

3.2. Option on Portfolio Loss

The payoff to many standard and exotic portfolio credit derivatives can be expressed in terms of a call option on the portfolio loss L^n, whose payoff at expiry T is $(L_T^n - q)^+$, where q is the strike price. To value such an option, let

$$Q(k, x) = -\frac{1}{2\pi} \int_{\gamma i - \infty}^{\gamma i + \infty} \frac{e^{-izx}}{z^2} (g(z))^k \, dz \tag{12}$$

for $k \in \mathbb{N}$ and $x \geqslant 0$. Here, the characteristic function g of the loss at default is extended to the complex plane. If S_g is the region of convergence of g, then γ is chosen such that $\gamma i \in S_g \cap \{z \in \mathbb{C}: \operatorname{Im}(z) < 0\}$. The next result shows that $Q(k, x)$ is the conditional expected payoff at t of an option on the loss over $(t, T]$ with strike x and maturity T, given that the number of defaults in $(t, T]$ equals k. In §4, we provide parametric models of the loss at default for which $Q(k, x)$ can be calculated explicitly.

Proposition 3.1. *The undiscounted price at time t of a call option on the portfolio loss with maturity $T \geqslant t$ and nonnegative strike q is given by*

$$E[(L_T^n - q)^+ \mid \mathcal{G}_t] = \sum_{k=0}^{n-N_t^n} P_{t,T}(k) Q(k, q - L_t^n).$$

The proof is in the appendix. Because the negative of the partial derivative of the expected option payoff with respect to the strike is equal to $P[L_T^n > q \mid \mathcal{G}_t]$, Proposition 3.1 also yields the conditional distribution of portfolio loss for all future dates. For $x \geqslant 0$, whenever the derivative is well defined, we get the formula

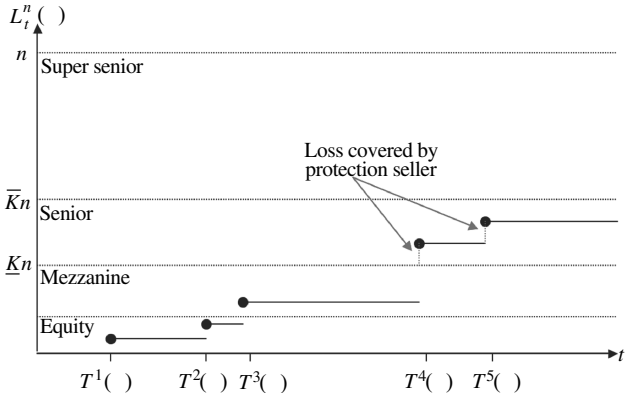
$$P[L_T^n - L_t^n \leqslant x \mid \mathcal{G}_t] = 1 + \sum_{k=0}^{n-N_t^n} P_{t,T}(k) \frac{\partial}{\partial x} Q(k, x). \tag{13}$$

3.3. Tranche and Index Swaps

Tranche and index swaps are bilateral financial contracts. One party buys insurance from the other party against the financial losses due to default in a portfolio of n bonds with common maturity date T, common quarterly payment dates (t_m), and common notional that we normalize to one.

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Figure 1. Paths of the portfolio loss process L^n and the ordered default times (T^k), and payments of the protection seller in a senior tranche swap.



Notes. The protection seller is exposed to the correlated default risk in the reference portfolio. The risk profile is determined by the location of the attachment points $\underline{K}$ and $\bar{K}$. The riskiest tranche is the equity tranche, for which $\underline{K} = 0$ and $\bar{K}$ is typically 3% or 10%. The least risky tranche is the super senior tranche, for which $\bar{K} = 1$ and $\underline{K}$ is typically 30% or 35%. This tranche bears losses only if all the other tranches are wiped out. The collection of all tranches is called a collateralized debt obligation (CDO).

The payments in a tranche swap are specified by a lower attachment point $\underline{K} \in [0, 1]$ and an upper attachment point $\bar{K} \in (\underline{K}, 1]$. The product of the difference $K = \bar{K} - \underline{K}$ and the portfolio notional n is the tranche notional. As illustrated in Figure 1, the protection seller covers portfolio losses as they occur, given that the cumulative losses are larger than $\underline{K}n$ but do not exceed $\bar{K}n$. The payments U_t at time t are equal to the cash flows to a portfolio of two call options written on the loss, both maturing at t:

$$U_t = (L_t^n - \underline{K}n)^+ - (L_t^n - \bar{K}n)^+. \tag{14}$$

The value of these payments at time $t \leqslant T$ is

$$D_{t,T} = e^{-r(T-t)}E[U_T \mid \mathcal{G}_t] - U_t + r\int_t^T e^{-r(s-t)}E[U_s \mid \mathcal{G}_t]\,ds. \tag{15}$$

The premium payments of the tranche protection buyer consist of two parts. The first part is an upfront payment, which is expressed as a fraction R of the tranche notional Kn. The second part is a stream of payments at dates (t_m). For a tranche with $\bar{K} < 1$, the cash flow at t_m is a fraction S of the difference between the tranche notional and the tranche loss U at t_m. Neglecting accruals, the value at time $t \leqslant T$ of these payments is

$$F_{t,T}(R,S) = RKn + S\sum_m e^{-r(t_m-t)}c_m(Kn - E[U_{t_m} \mid \mathcal{G}_t]), \tag{16}$$

where c_m is the day count fraction for period m. For a fixed upfront payment rate R, the fair tranche spread S is the solution $S = S_t(R)$ to the equation $D_{t,T} = F_{t,T}(R, S)$. Similarly, for a fixed tranche spread S, the fair tranche upfront rate R is the solution $R = R_t(S)$ to $D_{t,T} = F_{t,T}(R, S)$. Formulae (15) and (16) indicate that the fair tranche spread and upfront rate depend only on the values of call options on L_s^n with strikes $\underline{K}n$ and $\bar{K}n$ and maturities $s \in (t, T]$. The values of these calls are given by Proposition 3.1.

In an index swap, the protection seller bears a broader exposure to the correlated default risk in the underlying portfolio. The index protection seller covers all default losses that occur before the maturity of the swap. The value of these payments $D_{t,T}$ is given by formula (15) for $\underline{K} = 0$ and $\bar{K} = 1$. The index protection buyer makes periodic premium payments at the coupon dates (t_m). The cash flow at t_m is a fraction S of the total notional on the names that have survived until t_m. Neglecting premium accruals, the value at time $t \leqslant T$ of these cash flows is

$$F_{t,T}(S) = S\sum_m e^{-r(t_m-t)}c_m(n - E[N_{t_m}^n]). \tag{17}$$

The fair index swap spread S is the solution $S = S_t$ to the equation $D_{t,T} = F_{t,T}(S)$. It depends only on expected portfolio losses and defaults at future dates.

3.4. Tranche and Index Forwards and Options

Consider a tranche or index swap with maturity T. Suppose that the time t market spread is S and the upfront rate, if applicable, is R. For the buyer of protection, the *mark-to-market value* at time t of the swap is given by

$$M_{t,T}(R,S) = D_{t,T} - F_{t,T}(R,S). \tag{18}$$

This value is generally nontrivial if the market spread or upfront rate differ from the contract spread and upfront rate agreed on at inception of the swap. We consider a derivative on the mark-to-market value, which allows investors to express a view on the volatility of future spreads. The payoff to the derivative is of the form $h(M_{s,T}(R,S))$ for some payoff function $h\colon \mathbb{R} \to \mathbb{R}$ and expiration date $s < T$. The value at time $t \leqslant s$ of the derivative is given by the conditional expectation

$$e^{-r(s-t)}E[h(M_{s,T}(R,S)) \mid \mathcal{G}_t]. \tag{19}$$

In a *forward* the protection buyer agrees to enter the swap at some future date $s < T$ for a spread of S and an upfront rate of R. The payoff to the forward protection buyer at s is the mark-to-market value $M_{s,T}(R,S)$, so the payoff function h is the identity function. An *option* imparts the right to enter the swap as the protection buyer at some exercise date s and at strike spread S and upfront rate R. The corresponding payoff function is given by $h(x) = x^+$. Similarly, the payoff function of an option imparting the right to enter the swap as the protection seller is given by $h(x) = (-x)^+$.

A *cancelable swap* with maturity date T, exercise date $s < T$, spread S, and upfront rate R is a swap with the embedded option for one party to cancel the contract at s.

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If the buyer of protection is entitled to cancel the contract, the cancelable swap can be thought of as a standard swap on $[0, s]$ and an option to buy further protection on $[s, T]$. Hence, its value to the protection buyer at $t < s$ is given by the current mark-to-market value $M_{t,s}(R, S)$ plus the value of an option $e^{-r(s-t)}E[(M_{s,T}(R, S))^+ \mid \mathcal{G}_t]$. For a given upfront rate R, the fair spread at t is the value $S = S_t(R)$ that sets this value equal to zero.

To calculate the derivative price (19), we assume that the activity rate ν follows a Markov process. This assumption is not overly restrictive because most parametric activity rate models that support analytic solutions for the Laplace transform (7) are Markov. Under this assumption, the portfolio default probability $P_{t,s}(k)$ is a function of N_t and ν_t by Proposition 2.1. As a result, $E[(L_s^n - q)^+ \mid \mathcal{G}_t] = V_{t,s}^q(N_t^n, L_t^n, \nu_t)$ for some deterministic function $V_{t,s}^q$ given by Proposition 3.1. Now, note that $M_{t,T}(R, S)$ is a function of $E[U_s \mid \mathcal{G}_t]$ for all $s \in [t, T]$, where U is the call spread payoff (14). Hence,

$$M_{t,T}(R, S) = G_{t,T}(N_t^n, L_t^n, \nu_t; R, S)$$

for some function $G_{t,T}\colon \mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ given by

$$\begin{aligned} G_{t,T}&(k, x, z; R, S) \\ &= e^{-r(T-t)}\big(V_{t,T}^{\underline{K}n}(k, x, z) - V_{t,T}^{\bar{K}n}(k, x, z)\big) \\ &\quad - (x - \underline{K}n)^+ + (x - \bar{K}n)^+ \\ &\quad + r\int_t^T e^{-r(s-t)}\big(V_{t,s}^{\underline{K}n}(k, x, z) - V_{t,s}^{\bar{K}n}(k, x, z)\big)\,ds - RKn \\ &\quad - S\sum_{t < t_m \leqslant T} e^{-r(t_m - t)} \\ &\quad \cdot c_m\big(Kn - V_{t,t_m}^{\underline{K}n}(k, x, z) + V_{t,t_m}^{\bar{K}n}(k, x, z)\big). \end{aligned} \tag{20}$$

The time integral in this function can be replaced by a summation if we consider any losses given default to be paid at the following premium payment date t_m. This innocuous assumption leads to a significant simplification of the numerical calculation of G. For fixed contract parameters R, S, T, s, and payoff function h, letting $H(k, x, z) = h(G_{s,T}(k, x, z; R, S))$, the derivative price (19) is then

$$\begin{aligned} E[H(N_s^n, L_s^n, \nu_s) \mid \mathcal{G}_t] \\ = \int_0^\infty \int_0^\infty \sum_{k=0}^{n-N_t^n} H(k + N_t^n, x + L_t^n, y)\, d\mu_{t,s}(k, x, y)\, dy. \end{aligned}$$

where $\mu_{t,s}(k, x, y)$ is the joint conditional distribution function of the default count, loss, and activity rate given $\mathcal{G}_t$. Note that if the loss at default is a discrete random variable, then the inner integral becomes a sum that can be computed exactly. For parametric examples, see §4. If, moreover, the time change is deterministic (a case discussed in §4 as well), then the joint distribution is degenerate, and we can remove the outer integral with respect to dy. This last step will most likely produce the greatest reduction in computational effort; however, it also greatly reduces the flexibility of the model.

It remains to calculate $\mu_{t,s}(k, x, y)$. Using informal notation, by conditioning we get

$$\begin{aligned} \mu_{t,s}(k, x, y)\, dy &= P[N_s^n - N_t^n = k, L_s^n - L_t^n \leqslant x, \nu_s \in dy \mid \mathcal{G}_t] \\ &= P[L_s^n - L_t^n \leqslant x \mid N_s^n - N_t^n = k, \mathcal{G}_t] \\ &\quad \cdot P[N_s^n - N_t^n = k, \nu_s \in dy \mid \mathcal{G}_t] \end{aligned}$$

for $k \in \mathbb{N}$, $x > 0$, and $y > 0$. In the second line, we used the fact that conditional on the count, the loss is independent of the activity rate. From formula (13), it is clear that whenever the derivative is well defined,

$$P[L_s^n - L_t^n \leqslant x \mid N_s^n - N_t^n = k, \mathcal{G}_t] = 1 + \frac{\partial}{\partial x} Q(k, x).$$

Similarly to formula (10), we furthermore have that

$$\begin{aligned} P[N_s^n - N_t^n = k, \nu_s \in dy \mid \mathcal{G}_t] \\ = \begin{cases} P[N_s - N_t = k, \nu_s \in dy \mid \mathcal{G}_t] & \text{if } k < n - N_t^n, \\ P[N_s - N_t \geqslant k, \nu_s \in dy \mid \mathcal{G}_t] & \text{if } k = n - N_t^n, \\ 0 & \text{if } k > n - N_t^n, \end{cases} \end{aligned}$$

where the joint distribution of the time-changed birth process and the activity rate is expressed in formula (9) in terms of the extended Laplace transform.

3.5. Constituent Default Probabilities

Consider a position in a portfolio credit derivative such as an index or tranche swap. The mark-to-market value of the position (18) is sensitive to changes in the credit risk of any portfolio constituent firm. To estimate the sensitivities and construct the corresponding hedging strategies, we need to construct and calibrate constituent models that are compatible with the portfolio model.

We use *random thinning* to link the constituent default processes and the portfolio default process N^n. The idea is to allocate a fraction Z_t^j of the portfolio intensity λ_t^n to the constituent j. The value Z_t^j is the conditional probability at time t that constituent j is the next defaulter given that a default is imminent. Therefore, the sum of the Z_t^j over j must equal one unless all names in the portfolio are in default. If all constituents are in default, the thinning processes vanish.

Letting τ^j be the default time of firm j, Giesecke and Goldberg (2005) show that there is a predictable thinning process Z^j such that the conditional probability of firm j defaulting in the time period $(t, T]$ is given by

$$P[t < \tau^j \leqslant T \mid \mathcal{G}_t] = \int_t^T E[Z_s^j \lambda_s^n \mid \mathcal{G}_t]\, ds. \tag{21}$$

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Because the loss at default is independent of the default time, the constituent default probability (21) completely determines the price of a bond issued by firm j, the price for protection against the default of firm j, and, more generally, the price of a security referenced on firm j. Both the constituent probability and the portfolio default probability given in Proposition 2.1 are functions of the portfolio intensity λ^n. This relationship generates the link between the mark-to-market value of a portfolio derivative position and the mark-to-market value of a security referenced on a constituent name.

To obtain a simple parametric constituent model, we suppose that the thinning process Z^j is constant between events:

$$Z^j = p^j_{N^n_-}, \quad j = 1, 2, \ldots, n, \tag{22}$$

where p^j_k is the probability that name j is the $(k+1)$th defaulter for $k = 0, 1, \ldots, n$. Clearly, $p^j_n = 0$ for all j. Furthermore, the rows and columns of the matrix (p^j_k) must sum to one. Then, $Z^j\lambda^n = p^j_{N^n_-}\nu(c + \delta N^n)1_{\{N<n\}}$, so formula (21) gives

$$P[t < \tau^j \leqslant T \mid \mathcal{G}_t] = \sum_{k=0}^{n-N^n_t} p^j_{k+N^n_t} w_{t,T}(k), \tag{23}$$

where the weights

$$w_{t,T}(k) = (c + \delta(k + N^n_t)) \int_t^T E[\nu_s 1_{\{N^n_s - N^n_t = k\}} \mid \mathcal{G}_t]\, ds$$

do not depend on j. The value $w_{t,T}(k)$ represents the conditional probability at t that the $(N^n_t + k + 1)$th default is in the period $(t, T]$. Note that if all names in the portfolio have defaulted by time t, then the right side of Equation (23) is zero and so is the conditional default probability. To calculate the weights in formula (23), note that

$$E[\nu_s 1_{\{N^n_s - N^n_t = k\}} \mid \mathcal{G}_t] = \begin{cases} E[\nu_s 1_{\{N_s - N_t = k\}} \mid \mathcal{G}_t] & \text{if } k < n - N^n_t, \\ E[\nu_s 1_{\{N_s - N_t \geqslant k\}} \mid \mathcal{G}_t] & \text{if } k = n - N^n_t, \\ 0 & \text{if } k > n - N^n_t, \end{cases}$$

and from Proposition 2.2, for $t \leqslant s$ and $k \in \mathbb{N}$, whenever the derivative is well-defined,

$$E[\nu_s 1_{\{N_s - N_t = k\}} \mid \mathcal{G}_t] = -i\frac{\Gamma(C_t + k)}{\Gamma(C_t)k!} \sum_{m=0}^{k} \binom{k}{m} (-1)^m \frac{\partial}{\partial v} \mathcal{T}_{t,s}(\delta(m + C_t), v)\bigg|_{v=0}.$$

In §4 below, we consider parametric activity rate models for which the extended Laplace transform $\mathcal{T}_{t,s}(u, v)$ is analytically tractable. These activity rate models then generate analytically tractable constituent and portfolio default probabilities.

4. Parametric Model Specification

We illustrate the specification of the default and loss processes. We address three elements that are specified independently of each other: the birth process, the activity rate ν that generates the time change $T_t = \int_0^t \nu_s\, ds$, and the loss at default l_k.

4.1. Birth Process

The birth process controls the feedback of events on arrival rates. The sensitivity of arrival rates to an event is governed by the parameter δ. In case $\delta \to 0$, the birth process is a Poisson process with rate c. The independent time change generates *doubly stochastic Poisson processes*. Here, conditional on a realization of ν, events form a time-inhomogeneous Poisson process with time-varying intensity νc. There is no event feedback. If δ does not vanish, then the time change generates self-exciting *doubly stochastic birth processes*. Here, conditional on a realization of ν, arrivals form a time-inhomogeneous birth process with an intensity $\nu(c + \delta N)$ that varies stochastically due to its dependence on N.

4.2. Time Change

The time change generates stochastic variation in arrival rates between events. The conditional distribution of the time change is encoded in the extended Laplace transform $\mathcal{T}_{t,s}(u, v) = E[e^{-u\int_t^s \nu_x\, dx + iv\nu_s} \mid \mathcal{G}_t]$. The transform is analogous to the price at t of a security paying $\exp(iv\nu_s)$ at s, if short-term interest rates follow the process $u\nu$. This observation suggests the existing security-pricing model specifications as candidate specifications for the activity rate. In particular, deterministic, affine, and quadratic activity rate models guarantee computational tractability for the Laplace transform. Recall that if the Laplace transform is analytically tractable, then so are the distribution of the default process (10) and, if a suitable model for the loss at default is chosen, the distribution of the loss process (13) and the pricing of standard and exotic multiname credit derivatives.

4.2.1. Deterministic Activity Rate. Suppose that the activity rate is a deterministic function of time $\nu(t) > 0$ such that the integral (3) is finite and increasing to infinity. Then, the time change generates a *time-inhomogeneous birth process* N, whose intensity $\lambda_t = \nu(t)(c + \delta N_t)$ is an affine function of N_t with time-varying coefficients. The intensity has dynamics

$$d\lambda_t = (c + \delta N_t)\nu'(t)dt + \nu(t)\delta dN_t,$$

assuming that $\nu(t)$ has derivative $\nu'(t)$. The extended transform of the time change is

$$\mathcal{T}_{t,s}(u, v) = e^{-u\int_t^s \nu(x)\, dx + iv\nu(s)},$$

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and together with the binomial theorem, Proposition 2.1 shows that the conditional distribution of N is negative binomial:

$$P[N_s - N_t = k \mid \mathcal{G}_t] = \frac{\Gamma(C_t + k)}{\Gamma(C_t)k!}(\mathcal{T}_{t,s}(\delta, 0))^{C_t}(1 - \mathcal{T}_{t,s}(\delta, 0))^k;$$

compare with formula (2). Thus, the time change representation yields an alternative approach to Kendall's (1948) classical analysis of the time-inhomogeneous birth process.

Example 4.1 (Polya Process). Suppose that we would like to choose the rate at which the expected number of arrivals grows (i.e., linear, power, square root). For $\alpha, \beta > 0$, we take the activity rate $\nu(t) = \alpha\beta/(1 + \beta t)$. Then, the intensity satisfies

$$d\lambda_t = -\frac{\beta}{1+\beta t}\lambda_t\, dt + \frac{\delta}{\eta_t}\lambda_t\, dN_t,$$

where $\eta_t = c + \delta N_t$. The size of a feedback jump is proportional to the intensity, with a proportionality factor following the process δ/η. Because $\lambda/\eta = \nu$, the size of the jump at an event is inversely proportional to how long it takes for the event to occur. Between events, the intensity decays toward zero at a rate proportional to the parameter β and inversely proportional to time. The level at time t is increasing in the number of events observed to time t, but decreasing in the amount of time passed. We get

$$\mathcal{T}_{t,s}(u, 0) = \left(1 + \frac{\beta(s-t)}{1+\beta t}\right)^{-\alpha u},$$

so the conditional mean and variance of the number of arrivals are given by

$$E[N_s - N_t \mid \mathcal{G}_t] = \frac{C_t(1 - \mathcal{T}_{t,s}(\delta, 0))}{\mathcal{T}_{t,s}(\delta, 0)} \quad \text{and}$$

$$\mathrm{Var}[N_s - N_t \mid \mathcal{G}_t] = \frac{C_t(1 - \mathcal{T}_{t,s}(\delta, 0))}{(\mathcal{T}_{t,s}(\delta, 0))^2}.$$

When $\alpha = 1/\delta$, the mean increases linearly, and the variance increases quadratically. If additionally $\delta = \beta c$, then the time-changed birth process N is a *Polya process*; see Feller (1957, Chapter 17.11).

4.2.2. Simple Random Activity Rate. To introduce random effects into the activity rate in a simple and tractable way, suppose that $\nu_t = \nu(t)X$, where $\nu(t) > 0$ is a deterministic function of time and $X > 0$ is an independent random variable that represents a stochastic risk factor. Suppose that these quantities are chosen such that the integral (3) is finite and increasing to infinity, almost surely. This specification generates a time-changed birth process N that is a *mixed time-inhomogeneous birth process*. Note that in the special case where $\delta \to 0$ and $\nu(t) = 1$, N is a classical mixed Poisson process. In the general case, the intensity $\lambda_t = \nu(t)X(c + \delta N_t)$ is an affine function of N_t. Conditional on a realization of X, the coefficients are functions of time. Assuming that ν has derivative ν', the intensity has dynamics

$$d\lambda_t = (c + \delta N_t)\nu'(t)X\, dt + \nu(t)X\delta\, dN_t.$$

The extended transform of the time change is

$$\mathcal{T}_{t,s}(u, v) = \varphi\left(iu\int_t^s \nu(x)\, dx + v\nu(s)\right), \tag{24}$$

where, for $z \in \mathbb{C}$, the function $\varphi(z) = E[e^{izX}]$ is the generalized Fourier-Stieltjes transform of the distribution of X. For many specifications of the risk factor X, this transform can be calculated explicitly. Examples include a variety of continuous and discrete distributions, including the binomial, the Poisson, and the gamma distributions.

Together with formula (24), Proposition 2.1 gives an explicit expression of the distribution of a mixed time-inhomogeneous birth process N. Because conditional on a realization of the risk factor X, N is a time-inhomogeneous birth process whose distribution is negative binomial, and the distribution of N can be thought of as a mixture of negative binomial distributions. Mixed negative binomial distributions also arise in some of the models studied by Brigo et al. (2006). They consider linear combinations of gamma-mixed Poisson processes, each of which has a negative binomial marginal distribution.

4.2.3. Affine Activity Rate. Consider a vector X of risk factors. Suppose that X is an affine jump diffusion in the sense of Duffie et al. (2000). This means that X is a Markov process in some state space $D \subset \mathbb{R}^d$ that is a strong solution to the stochastic differential equation

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t + dJ_t,$$

where W is an $\mathbb{R}^d$-valued standard Brownian motion, $\mu\colon D \to \mathbb{R}^d$ is the drift, $\sigma\colon D \to \mathbb{R}^{d\times d}$ is the volatility, and J is an $\mathbb{R}^d_+$-valued point process whose events arrive with common intensity $h(X)$ where $h\colon D \to [0, \infty)$, and whose jump sizes are drawn from a fixed distribution F on $\mathbb{R}^d_+$ that has no mass at zero. The processes W and J are independent of the birth process. To complete the specification of X, assume that

$$\mu(x) = K_0 + K_1 x, \quad K_0 \in \mathbb{R}^d,\ K_1 \in \mathbb{R}^{d\times d},$$

$$(\sigma(x)\sigma(x)^\top)_{jk} = (H_0)_{jk} + (H_1)_{jk} \cdot x, \quad H_0 \in \mathbb{R}^{d\times d},\ H_1 \in \mathbb{R}^{d\times d\times d},$$

$$h(x) = M_0 + M_1 \cdot x, \quad M_0 \in \mathbb{R},\ M_1 \in \mathbb{R}^d.$$

The activity rate follows the multifactor jump-diffusion model $\nu = \nu(X)$, where $\nu\colon D \to [0, \infty)$ is an affine function

$$\nu(x) = R_0 + R_1 \cdot x, \quad R_0 \in \mathbb{R},\ R_1 \in \mathbb{R}^d,$$

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such that $\nu_t > 0$ and the integral (3) is finite and increasing to infinity, almost surely. This activity rate specification generates a time-changed birth process N whose intensity $\lambda_t = (R_0 + R_1 \cdot X_t)(c + \delta N_t)$ is an affine function of N_t with stochastically varying coefficients that are driven by X. Therefore, the intensity follows a jump-diffusion process with feedback jumps from default:

$$d\lambda_t = (c + \delta N_{t-})R_1 \cdot dX_t + (R_0 + R_1 \cdot X_{t-})\delta dN_t, \tag{25}$$

so that the drift and volatility of the intensity have affine dependence on N. The specification (25) demonstrates that the time-changed birth processes generated by an affine activity rate model form a strict subset of the class of affine point processes introduced by Errais et al. (2006). The intensity of an affine point process may have nontrivial drift and diffusion terms that are independent of the feedback term. Any nontrivial drift and diffusion terms in the intensity of a self-exciting time-changed birth process that is generated by an affine activity rate must depend on N.[5] Further, the feedback term in the intensity of an affine point process may depend on the loss at default, allowing one to model a dependence structure among default and recovery rates. In the specification (25), this dependence structure is trivial.

Duffie et al. (2000) give technical regularity conditions guaranteeing that

$$\mathcal{J}_{t,s}(u, v) = e^{ivR_0 + \alpha(t) + \beta(t) \cdot X_t}, \tag{26}$$

where the coefficient functions $\alpha(t) = \alpha(u, v, t, s)$ and $\beta(t) = \beta(u, v, t, s)$ satisfy the complex-valued ordinary differential equations

$$\partial_t \beta(t) = uR_1 - K_1^\top \beta(t) - \tfrac{1}{2}\beta(t)^\top H_1 \beta(t) - M_1(\theta(\beta(t)) - 1),$$
$$\partial_t \alpha(t) = uR_0 - K_0 \cdot \beta(t) - \tfrac{1}{2}\beta(t)^\top H_0 \beta(t) - M_0(\theta(\beta(t)) - 1),$$

with boundary conditions $\alpha(s) = 0$ and $\beta(s) = ivR_1$ and jump transform

$$\theta(z) = \int_{\mathbb{R}_+^d} e^{z \cdot c}\, dF(c), \quad z \in \mathbb{C}.$$

If the activity rate is affine in X, then the time change approach leads to intensity specifications that are narrower than those studied by Errais et al. (2006). On the other hand, the time change approach generates significant computational benefits relative to the Fourier-transform based approach developed in Errais et al. (2006) for affine point processes. Here is an affine example that is especially tractable.

Example 4.2 (Square Root Activity Rate). Suppose that $K_0 = \kappa\bar{\nu}$, $K_1 = -\kappa$, $H_0 = 0$, and $H_1 = \sigma^2$ for κ, $\bar{\nu}$, and σ positive constants such that $2\kappa\bar{\nu} > \sigma^2$. Further, let $R_0 = 0$, $R_1 = 1$, and $X_0 > 0$ so that $\nu = X$ follows the mean-reverting diffusion process

$$d\nu_t = \kappa(\bar{\nu} - \nu_t)\, dt + \sigma\sqrt{\nu_t}\, dW_t, \tag{27}$$

which is strictly positive almost surely. This specification generates the intensity dynamics

$$d\lambda_t = \kappa(\bar{\nu}\eta_t - \lambda_t)\, dt + \sigma\sqrt{\eta_t \lambda_t}\, dW_t + \frac{\delta}{\eta_t}\lambda_t dN_t, \tag{28}$$

where $\eta_t = c + \delta N_t$. The size of a feedback jump is proportional to the intensity, with a proportionality factor following the process δ/η. Because $\lambda/\eta = \nu$, the size of the intensity jump at an event is governed by the independent square-root diffusion (27). Between events, the intensity reverts to the level $\mu\eta_t$ at rate κ. The level increases in the number of events. The diffusive fluctuation of the intensity between events is driven by a Brownian motion. The diffusive volatility follows a stochastic process. At any time the instantaneous variance is proportional to the intensity, with proportionality factor given by $\sigma^2\eta_t$. Thus, the diffusive volatility at t is an increasing function of N_t. The results in Duffie and Garleanu (2001) imply that the transform coefficients $\alpha(t)$ and $\beta(t)$ in formula (26) satisfy

$$\beta(t) = \frac{1 + ae^{b(s-t)}}{c + de^{b(s-t)}},$$
$$\alpha(t) = \frac{\kappa\bar{\nu}(ac - d)}{bcd}\log\frac{c + de^{b(s-t)}}{c + d} + \frac{\kappa\bar{\nu}}{c}(s - t),$$

with

$$c = \frac{\kappa + \sqrt{\kappa^2 + 2\sigma^2 u}}{-2u},$$
$$d = (1 - civ)\frac{-\kappa + \sigma^2 iv + \sqrt{\kappa^2 + 2\sigma^2 u}}{-2\kappa iv + \sigma^2(iv)^2 - 2u},$$
$$a = (d + c)iv - 1,$$
$$b = \frac{d(-\kappa - 2uc) + a(-\kappa c + \sigma^2)}{ac - d}.$$

4.2.4. Quadratic Activity Rate. Again, consider a vector X of risk factors. Suppose that X is a Markov process in some state space $D \subset \mathbb{R}^d$ that is a strong solution to the stochastic differential equation

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t,$$

where W is an $\mathbb{R}^d$-valued standard Brownian motion that is independent of the birth process, $\mu : D \to \mathbb{R}^d$ is the drift, and $\sigma : D \to \mathbb{R}^{d\times d}$ is the volatility. Further, X is a scaled and translated Ornstein-Uhlenbeck process:

$$\mu(x) = K_0 + K_1 x, \quad K_0 \in \mathbb{R}^d,\ K_1 \in \mathbb{R}^{d\times d},$$
$$\sigma(x)\sigma(x)^\top = H_0, \quad H_0 \in \mathbb{R}^{d\times d}.$$

The activity rate follows the multifactor diffusion model $\nu = \nu(X)$, where $\nu : D \to [0, \infty)$ is a quadratic function:

$$\nu(x) = R_0 + R_1 \cdot x + R_2 x \cdot x, \quad R_0 \geqslant 0,\ R_1 \in \mathbb{R}^d,\ R_2 \in \mathbb{R}^{d\times d},$$

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such that $\nu_t > 0$ and the integral (3) is finite and increasing to infinity, almost surely. A sufficient condition for $\nu_t > 0$ is to let R_2 be positive definite and $R_0 > \frac{1}{4}R_2R_1 \cdot R_1$. This activity rate generates a time-changed birth process N whose intensity $\lambda_t = (R_0 + R_1 \cdot X_t + R_2X_t \cdot X_t)(c + \delta N_t)$ is an affine function of N_t with stochastically varying coefficients that are driven by X. The intensity follows a jump-diffusion process with feedback jumps,

$$d\lambda_t = (c + \delta N_{t-})(R_1 \cdot dX_t + d(R_2X_t \cdot X_t)) + (R_0 + R_1 \cdot X_t + R_2X_t \cdot X_t)\delta dN_t,$$

so that the drift and volatility of the intensity have affine dependence on N. Chen et al. (2002) and Leippold and Wu (2002) give technical conditions guaranteeing that

$$\begin{aligned}\mathcal{T}_{t,s}(u,v) &= E[e^{-u\int_t^s R(X_x)\,dx + ivR(X_s)} \mid \mathcal{G}_t] \\ &= e^{ivR_0}E[e^{-u\int_t^s R(X_x)\,dx + ivR_1\cdot X_s + ivR_2X_s\cdot X_s} \mid \mathcal{G}_t] \\ &= e^{ivR_0 + \alpha(t) + \beta(t)\cdot X_t + \gamma(t)X_t\cdot X_t},\end{aligned}$$

where the coefficient functions $\alpha(t) = \alpha(u, v, t, s)$, $\beta(t) = \beta(u, v, t, s)$, and $\gamma(t) = \gamma(u, v, t, s)$ satisfy the complex-valued ordinary differential equations

$$\begin{aligned}\partial_t\gamma(t) &= uR_2 - 2\gamma(t)K_1 - 2\gamma(t)H_0\gamma(t), \\ \partial_t\beta(t) &= uR_1 - 2\gamma(t)K_0 - K_1^\top\beta(t) - 2\gamma(t)H_0^\top\beta(t), \\ \partial_t\alpha(t) &= uR_0 - \beta(t)^\top K_0 - \tfrac{1}{2}\beta(t)^\top H_0\beta(t) - \operatorname{tr}[H_0\gamma(t)],\end{aligned}$$

with boundary conditions $\alpha(s) = 0$, $\beta(s) = ivR_1$, and $\gamma(s) = ivR_2$. Chen et al. (2002) give explicit solutions to these equations for the case $d = 1$. This leads to a closed-form expression for the extended transform $\mathcal{T}_{t,s}$.

4.2.5. Mixtures. In the multifactor affine and quadratic models discussed above, the activity rate is driven by a family of risk factors. In the same spirit, we can consider a mixture

$$\nu_t = \sum_{k=1}^m w^k\nu_t^k$$

of m independent activity rates ν^k, where the w^k are constant weights. For example, the ν^k could be deterministic, affine, or quadratic multifactor models on their own. If the weights are unit valued and add to one, then we can think of ν being equal to ν^k with probability w^k. The corresponding extended Laplace transform $\mathcal{T}_{t,s}(u, v)$ can then be computed as the product of the extended Laplace transforms of the individual time changes.

Example 4.3. Suppose that $\nu = \nu^1 + \nu^2$, where ν^1 follows a square-root diffusion as in Example 4.2 and $\nu^2 = \kappa h$ as in Example 4.1, where h is a strictly positive, continuously differentiable, and locally integrable deterministic function of time. Then,

$$d\nu_t^1 = \kappa(\bar\nu - \nu_t^1)\,dt + \sigma\sqrt{\nu_t^1}\,dW_t, \quad d\nu_t^2 = \kappa h'(t)\,dt.$$

This specification generates a time-changed birth process with intensity dynamics

$$d\lambda_t = \kappa(\eta_t(\bar\nu + h'(t) + \kappa h(t)) - \lambda_t)\,dt + \sigma\sqrt{\eta_t(\lambda_t - \kappa h(t)\eta_t)}\,dW_t + \frac{\delta}{\eta_t}\lambda_t\,dN_t,$$

where $\eta_t = c + \delta N_t$. These dynamics are similar to the dynamics (28) in Example 4.2, with one significant difference. Between arrivals, the intensity reverts to the stochastically varying level $\eta_t(\bar\nu + h'(t) + \kappa h(t))$ with rate κ. In particular, if we choose h to be decreasing, then the event-induced upward trend coming from η is counteracted. A similar effect occurs for the diffusion coefficient.

4.3. Loss at Default

The loss at default l_k is specified independently of the time change. Its distribution does not depend on k and is encoded in the generalized Fourier-Stieltjes transform $g(z) = E[e^{izl_1}]$, $z \in \mathbb{C}$, which is the characteristic function of l_1 if we restrict z to be real. Because the variables l_k are independent, the generalized Fourier-Stieltjes transform of the total loss associated with k defaults is given by the kth power of $g(z)$. From formula (12), the conditional expected payoff at t of an option on the loss over $(t, s]$ with strike x and maturity s, given that the number of defaults in $(t, s]$ equals k, is

$$Q(k,x) = -\frac{1}{2\pi}\int_{\gamma i-\infty}^{\gamma i+\infty}\frac{e^{-izx}}{z^2}(g(z))^k\,dz.$$

If Q is analytically tractable along with the extended Laplace transform of the time change $\mathcal{T}_{t,s}$, then so are all the results provided in this paper. We consider several specifications of the loss at default that lead to closed-form expressions for Q.

4.3.1. Binary Distribution. Suppose that the loss at default equals $d_1 \geqslant 0$ with probability $p \in [0, 1]$ and $d_2 > d_1$ with probability $1 - p$. Then, the generalized Fourier-Stieltjes transform of the distribution of l_k has three parameters and is given by

$$g(z) = pe^{izd_1} + (1-p)e^{izd_2}, \quad z \in \mathbb{C}.$$

Note that the region of convergence of $g(z)$ is the entire complex plane. Letting $a(k, j, x) = jd_1 + (k - j)\,d_2 - x$, and choosing γ such that $\gamma i \in \{z \in \mathbb{C}\colon \operatorname{Im}(z) < 0\}$, we get

$$\begin{aligned}Q(k,x) &= -\frac{1}{2\pi}\int_{\gamma i-\infty}^{\gamma i+\infty}\sum_{j=0}^k\binom{k}{j}p^j(1-p)^{k-j}e^{iza(k,j,x)}\frac{1}{z^2}\,dz \\ &= \sum_{j=0}^k\binom{k}{j}p^j(1-p)^{k-j}a(k,j,x)1_{\{a(k,j,x)\geqslant 0\}},\end{aligned}$$

which admits a convenient derivative as well.

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4.3.2. Binomial Distribution. Suppose that the loss at default has a scaled binomial distribution with parameters b and p and support in $\{0, d/b, 2d/b, \ldots, (b-1)\,d/b, d\}$ for some positive integer d. The corresponding generalized Fourier-Stieltjes transform has three parameters, and is given by

$$g(z) = (pe^{izd/b} + 1 - p)^b, \quad z \in \mathbb{C}.$$

The region of convergence of $g(z)$ is the entire complex plane. Letting $a(j,x) = jd/b - x$ and choosing γ such that $\gamma i \in \{z \in \mathbb{C}\colon \mathrm{Im}(z) < 0\}$, we calculate

$$\begin{aligned} Q(k,x) &= -\frac{1}{2\pi}\int_{\gamma i-\infty}^{\gamma i+\infty} \sum_{j=0}^{bk} \binom{bk}{j} p^j (1-p)^{bk-j} e^{iza(j,x)} \frac{1}{z^2}\, dz \\ &= \sum_{j=0}^{bk} \binom{bk}{j} p^j (1-p)^{bk-j} a(j,x) 1_{\{a(j,x)\geqslant 0\}}, \end{aligned}$$

which also has a convenient derivative.

4.3.3. Gamma Distribution. Suppose that the loss at default has a gamma distribution with shape parameter b valued in the positive integers and scale parameter $d > 0$. For $b = 1$, the gamma distribution is the exponential distribution. The corresponding generalized Fourier-Stieltjes transform is specified by two parameters and is given by

$$g(z) = (1 - izd)^{-b}, \quad z \in \mathbb{C}.$$

The region of convergence of $g(z)$ is $\{z \in \mathbb{C}\colon \mathrm{Im}(z) > -1/d\}$. Letting $a = x/d$, choosing γ such that $\gamma i \in \{z \in \mathbb{C}\colon -1/d < \mathrm{Im}(z) < 0\}$, and calculating the residue of the integrand at the singularity $z = -i/d$ as

$$\begin{aligned} &\mathrm{Res}[(1-idz)^{-bk} e^{-izx}, z = -i/d] \\ &\quad = -ide^{-a} \sum_{j=0}^{bk-1} \frac{(bk-j)a^j}{j!} \\ &\quad = -\frac{id}{(bk)!}(e^{-a}a^{bk+1} - (a-bk)\Gamma(bk+1,a)), \end{aligned}$$

where $\Gamma(b,x) = \int_x^\infty t^{b-1}e^{-t}\,dt$ is the incomplete gamma function, we get

$$\begin{aligned} &Q(k,x) \\ &\quad = \frac{d}{(bk)!}\big(e^{-x/d}(x/d)^{bk+1} - (x/d - bk)\Gamma(bk+1, x/d)\big). \end{aligned}$$

Also, the derivative admits a simple expression.

5. Market Calibration

We illustrate our results by implementing and calibrating an index and tranche swap, forward, and option pricer. We adopt the affine model specification of Example 4.2, where the activity rate ν follows a Feller diffusion with parameters ν_0, $\bar{\nu}$, κ, and σ. Following market practice, we assume that the loss at default l_k is fixed at 0.6. The parameter vector is $\theta = (\nu_0, \bar{\nu}, \kappa, \sigma, c, \delta)$, where c and δ are the parameters of the birth process.

We calibrate the model parameters to market rates of index and tranche swaps referenced on the CDX High Yield index portfolio of $n = 100$ North American constituents. The rates were observed on 5/11/2007 for maturities of five and seven years, and attachment points 0, 10, 15, 25, and 35%. Thus, along with the index quote, we have five quotes per maturity. The 0%–10% and 10%–15% tranches are quoted in terms of an upfront rate. The other tranches are quoted in terms of a quarterly spread. The risk-free rate $r = 0.05$. The reference date is $t = 0$.

We numerically solve the constrained nonlinear optimization problem

$$\min_{\theta\in\Theta} \sum_k \frac{(\mathrm{MarketMid}(k) - \mathrm{Model}(k,\theta))^2}{\mathrm{MarketMid}(k)}$$

$$\text{subject to} \quad 2\kappa\bar{\nu} \geqslant \sigma^2,$$

where $\Theta = [0,5]^3 \times [0,1] \times [0,5] \times [0,2]$ and the sum ranges over the quotes. The market midquote $\mathrm{MarketMid}(k)$ is the arithmetic average of the observed bid and ask quotes for index or tranche k. The model value $\mathrm{Model}(k,\theta)$ is given by the index and tranche pricing formulae derived in §3.3. These formulae, as well as the optimization, were implemented in Matlab and C++. In formula (20), we approximated the time integral based on quarterly time steps that coincide with the premium payment dates of the swap contracts. This approximation is consistent with standard industry practice. To evaluate the transform derived in Proposition 2.2 and formula (9), we used the high precision library GMP/MPFR. The computations were performed on a desktop PC with a 1.6 GHz Dual Core Intel processor and 2 GB of RAM.

We perform two different calibrations, each based on 100 sets of random initial values drawn from a uniform distribution over the parameter space Θ. In a first calibration, we fit to the rates of 5Y contracts, which are the most liquid. We assume that $\bar{\nu} = \nu_0$, leaving us with five parameters to fit five data points. The data and results are summarized in Table 1. The model fits the data very well, with an average absolute percentage error (AAPE) of 0.62%. In a second calibration, we let $\bar{\nu}$ vary freely and fit to both 5 and 7Y data (10 data points). The AAPE is higher at 3.28% and the optimal parameter values are

$$\begin{aligned} &\nu_0 = 0.186, \quad \bar{\nu} = 1.546, \quad \kappa = 0.566, \\ &\sigma = 1.199, \quad c = 3.751, \quad \delta = 0.005. \end{aligned} \tag{29}$$

The quality of these fits is roughly in line with those obtained by Giesecke and Kim (2007) for the affine point process model of Errais et al. (2006). Figure 2 shows the model implied loss distribution for different horizons.

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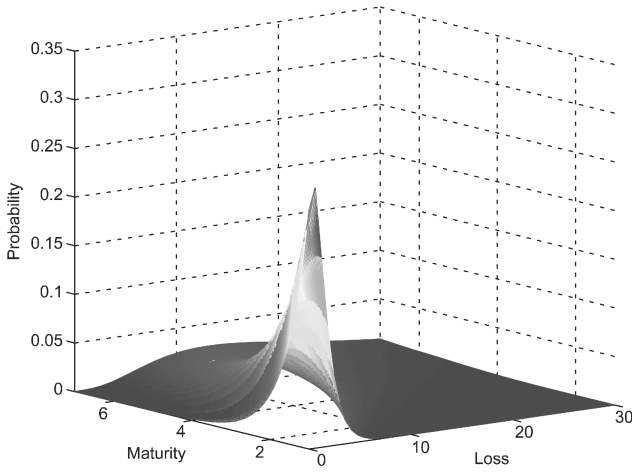
Table 1. Calibration data and results.

Maturity	Contract (%)	MarketBid	MarketAsk	5Y Fit	5 + 7Y Fit
5Y	0–10	70.50%	70.75%	71.88%	73.75%
	10–15	34.25%	34.50%	34.11%	33.85%
	15–25	316.00	319.00	317.87	309.37
	25–35	79.00	81.00	80.01	78.28
Index		262.85	263.10	261.93	268.30
7Y	0–10	80.13%	80.38%		82.71%
	10–15	55.50%	55.75%		54.30%
	15–25	582.00	587.00		563.30
	25–35	180.00	183.00		200.44
Index		307.50	307.75		305.87
AAPE				0.62%	3.28%

Notes. The CDX North America High Yield index and tranche swap market bid and ask rates were observed on 5/11/2007. The 0%–10% and 10%–15% tranches are quoted in terms of a percentage upfront rate. All other tranches and the index are quoted in terms of a quarterly spread that is measured in basis points. The column "5Y Fit" represents the calibration to 5Y data. The column "5+7Y Fit" represents the calibration to 5 and 7Y data. We report the average absolute percentage error AAPE.

The calibration illustrates a significant conceptual advantage of our modeling approach over the static copula models that are widely used in the financial industry. We specify the stochastic point process for the portfolio loss in terms of a set of parameters. The parameters govern index and tranche prices for all possible attachment points and maturities. Copula models, in contrast, are static. They specify the portfolio loss at a single horizon, making it difficult to capture the variation of market index and tranche swap rates across maturities and attachment points. Model values for tranches not used in the calibration are found by interpolation among parameter values, which is dubious because it may introduce arbitrage opportunities. Our approach allows us to avoid such procedures. Once calibrated, our model yields arbitrage-free prices of tranches for all maturities and all attachment points.

Figure 2. Loss distributions for different horizons implied by the model calibrated to 5 + 7Y CDX North America High Yield index and tranche rates.



Note. The distributions are kernel smoothed.

We can draw several conclusions from the parameter estimates in (29). The optimal parameter value that we obtain for the feedback control parameter δ are positive, indicating that the CDX market is sensitive to feedback from events. We also find a nontrivial value for the diffusive volatility σ, indicating the sensitivity of the market to a common factor that is represented by the Brownian motion W driving the time change. More extensive calibrations are required to shed light on the persistency of these effects through different calibration dates.

The mark-to-market value (18) of a swap position jumps at events. At an event, the mark-to-market value is adjusted to account for the realized loss. At the same time, its future time evolution is adjusted due to the feedback of the event on arrival rates. The Brownian motion W generates diffusive fluctuations in the mark-to-market value between events, reflecting random movements in the economic environment that affect the portfolio constituents. The time evolution of the mark-to-market value is meaningful and described by a concise set of economically significant parameters. The mark-to-market value is the underlying of a forward or option on the swap, and its dynamics are important because they govern the valuation of these contracts. The time evolution of the mark-to-market value implied by the static copula models is not meaningful. Therefore, these models are inappropriate for the pricing of exotic multiname derivatives.

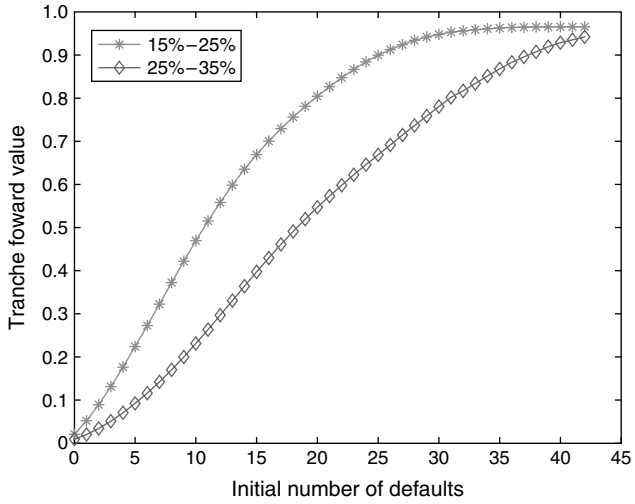
Next, we use the calibrated model (29) to price forwards and options on a 5Y tranche swap. We consider contracts that expire in six months and impart the right or the obligation to enter the swap as protection buyer. Figure 3 plots the value of forwards on the 15%–25% and 25%–35% tranches for each of several values of the current portfolio default count N_0^n. As forward spread S, we use the current 5Y midmarket tranche spread from Table 1. The forward value is increasing in N_0^n. Figure 4 shows the value of options on the 15%–25% and 25%–35% tranches for each of several values of the strike spread S. To plot the option values on the same scale, we normalize the strike spread by the current 5Y midmarket tranche spread from Table 1. The option values are decreasing in the strike. The option to buy protection on the more senior tranche is worth less than the option on the more junior tranche.

6. Conclusion

In this paper, we introduce a new family of stochastic point processes that can serve as probabilistic models for correlated event timing. The processes are obtained by time changing a classical birth process. They have attributes that are relevant in many applications. The self-exciting property captures the impact of an event on current and

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Figure 3. Value of forward contracts on the five-year 15%–25% and 25%–35% tranches for each of several values of the current portfolio default count N_0^n.

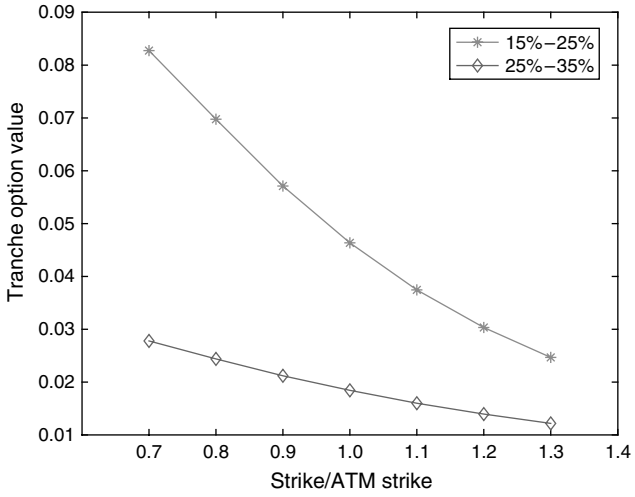


Note. The forwards expire in six months.

future event arrival rates. The random variation of arrival rates between events captures the stochastic dynamics of the environment. The processes are also computationally tractable: we derive a closed-form expression for their conditional distribution in terms of the Laplace transform of the time change, and we show how to explicitly calculate that Laplace transform for a wide range of parametric model specifications.

We use a time-changed birth process to model correlated corporate default risk. In this application, the process describes the arrival of unpredictable default events such as the failure of a firm to pay interest or principal on a loan or bond. Default events are correlated because firms are exposed to common economic risk factors such as interest rates or commodity prices. The random movement of these factors generates stochastic variation in default arrival rates. Due to the complex web of business, legal, and informational relationships in the economy, a default can also have a direct impact on the prospects of the surviving firms. The self-exciting property of a time-changed birth process captures this feedback. We study the valuation of multiname credit derivatives, which are financial securities that allow investors to trade correlated default risk. Through these derivatives, an investor can transfer risk to the capital markets. Our results on the time-changed birth process allow us to derive analytically tractable pricing relations for a range of multiname derivatives valuation problems. We illustrate our results by calibrating a time-changed birth process model to market prices of index and tranche swaps, which are standard multiname derivatives that are very liquid. We obtain accurate fits to the data. The calibrated model is then used to price exotic multiname derivatives. The results indicate the significance of event feedback and stochastic variation of the interevent intensity for the valuation of these exotic derivatives.

Figure 4. Value of option contracts on the five-year 15%–25% and 25%–35% tranches.



Note. The options expire in six months.

There is a range of other potential applications in financial engineering, as well as in other areas. In market microstructure analysis, a time-changed birth process can serve as a model for the arrival of security trades, which have been shown to cluster in time. In insurance, the process can describe the arrival of claims. In reliability, the process can model the failure of interdependent system components, or security breaches in a network. In health care, the process can describe the spread of diseases. In all these applications, event feedback and random influences on event arrival rates play a significant role, making a time-changed birth process ideally suited for their analysis.

Appendix. Proofs

Proof of Proposition 2.2. For fixed t and $s \geqslant t$, as well as fixed $u \geqslant 0$, $v \in \mathbb{R}$, let $r = -C_t$ and define the random variables $X = e^{\delta(T_s - T_t)} \in [1, \infty)$ and $Y = e^{iv\nu_s - u(T_s - T_t)}$ (so that $|Y| \leqslant 1$). By iterated expectation, conditioning on $T_s - T_t$ and ν_s, we get

$$\begin{aligned} f(w) &:= E[e^{-u(T_s-T_t)+iv\nu_s+iw(N^0_{T_s}-N^0_{T_t})} \mid \mathcal{G}_t] \\ &= E[e^{iv\nu_s-u(T_s-T_t)}(e^{iw}+(1-e^{iw})e^{\delta(T_s-T_t)})^r \mid \mathcal{G}_t] \\ &= E[Y(e^{iw}+X(1-e^{iw}))^r \mid \mathcal{G}_t]. \end{aligned}$$

By definition,

$$f(w) = \sum_{k=0}^{\infty} e^{iwk} E[Y 1_{\{N_s - N_t = k\}} \mid \mathcal{G}_t].$$

Because f is a periodic function with period 2π, we recognize the following Fourier series expansion pair:

$$f(w) = \sum_{k=-\infty}^{\infty} c_k e^{ikw} \quad \text{and} \quad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(w) e^{-ikw}\, dw.$$

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This shows that $c_k = E[Y1_{\{N_s-N_t=k\}} \mid \mathcal{G}_t]$ for $k \geqslant 0$ and $c_k = 0$ otherwise. Thus,

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} E[Y(e^{iw}+X(1-e^{iw}))^r \mid \mathcal{G}_t]e^{-ikw}\,dw$$
$$= \frac{1}{2\pi}E\left[Y\int_{-\pi}^{\pi} e^{-ikw}(e^{iw}+X(1-e^{iw}))^r\,dw \,\middle|\, \mathcal{G}_t\right]$$

by Fubini's theorem, which is justified because

$$E\left[\int_{-\pi}^{\pi} |Ye^{-ikw}(e^{iw}+X(1-e^{iw}))^r|\,dw \,\middle|\, \mathcal{G}_t\right]$$
$$\leqslant E\left[\int_{-\pi}^{\pi} |X+(1-X)e^{iw}|^r\,dw \,\middle|\, \mathcal{G}_t\right]$$
$$\leqslant E\left[\int_{-\pi}^{\pi} |X-|(1-X)e^{iw}||^r\,dw \,\middle|\, \mathcal{G}_t\right]$$
$$\leqslant E\left[\int_{-\pi}^{\pi} 1\,dw \,\middle|\, \mathcal{G}_t\right] < \infty.$$

Now we turn back to the computation of c_k. Put $z = e^{iw}$ and integrate counterclockwise along the unit circle to obtain

$$c_k = \frac{-i}{2\pi}E\left[Y\oint_{|z|=1}(z+X(1-z))^r z^{-k-1}dz \,\middle|\, \mathcal{G}_t\right]$$
$$= \frac{-i}{2\pi}E\left[Y\mathrm{Res}\left[\left(1-\frac{X-1}{X}z\right)^r X^r z^{-k-1},\ z_0=0\right] \,\middle|\, \mathcal{G}_t\right]$$
$$= \begin{cases} E[Y(-1)^k\binom{r}{k}(X-1)^k X^{r-k} \mid \mathcal{G}_t], & k \geqslant 0,\\ 0, & \text{else,}\end{cases}$$

where $\binom{r}{k} = r(r-1)\cdots(r-k+1)/k!$ is the generalized binomial coefficient. Thus,

$$E[e^{-u(T_s-T_t)+iv\nu_s}1_{\{N_s-N_t=k\}} \mid \mathcal{G}_t]$$
$$= c_k = E\left[Y(-1)^k\binom{r}{k}(X-1)^k X^{r-k} \,\middle|\, \mathcal{G}_t\right]$$

for $k \geqslant 0$, which can be simplified to

$$E[e^{-u(T_s-T_t)+iv\nu_s}1_{\{N_s-N_t=k\}} \mid \mathcal{G}_t]$$
$$= (-1)^k\binom{r}{k}\sum_{m=0}^{k}\binom{k}{m}(-1)^m E[YX^{r-m} \mid \mathcal{G}_t]$$

by applying the binomial theorem. Note that because $YX^{r-m} = e^{-(u-\delta(r-m))(T_s-T_t)+iv\nu_s}$, we have

$$E[YX^{r-m} \mid \mathcal{G}_t] = \mathcal{T}_{t,s}(u+\delta(m+C_t), v)$$

Lastly, by using the identity $(-1)^k\binom{r}{k} = \Gamma(-r+k)/\Gamma(-r)k!$, we get the result. □

Proof of Proposition 3.1. By conditioning on the number of defaults in $(t, s]$ and exploiting the independence of the loss at default variables l_i, we obtain

$$E[(L_s^n - L_t^n - q)^+ \mid \mathcal{G}_t]$$
$$= \sum_{k=0}^{n-N_t^n} P_{t,s}(k)E[(Y^k-q)^+ \mid \mathcal{G}_t], \tag{30}$$

where $t \leqslant s$ and $Y^k = l_1 + \cdots + l_k$. Because the l_i are independent of each other and of the information in $\mathcal{G}_t$, the *generalized* conditional Fourier-Stieltjes transform of the distribution F of Y^k is given by

$$E[e^{izY^k} \mid \mathcal{G}_t] = \int_{-\infty}^{\infty} e^{izx}\,dF(x) = (g(z))^k, \quad z \in S_g,$$

where the Fourier-Stieltjes transform g of the distribution of l_1 is extended to the complex plane, and S_g is the region of convergence of g. Similarly, the generalized Fourier transform of the option payoff is

$$\int_{-\infty}^{\infty} e^{izx}(x-q)^+\,dx = -\frac{e^{izq}}{z^2}, \quad z \in \mathbb{C},\ \mathrm{Im}(z) > 0.$$

Now Parseval's identity for generalized Fourier-Stieltjes transforms gives the result

$$E[(Y^k-q)^+ \mid \mathcal{G}_t] = \int_{-\infty}^{\infty}(x-q)^+\,dF(x)$$
$$= -\frac{1}{2\pi}\int_{\gamma i-\infty}^{\gamma i+\infty}\frac{e^{-izq}}{z^2}(g(z))^k\,dz,$$

where γ is chosen such that $\gamma i \in S_g \cap \{z \in \mathbb{C}: \mathrm{Im}(z) < 0\}$; see Lewis (2001) and Lee (2004) for a similar argument. □

Endnotes

1. Feller (1957) derives the differential equations governing the probabilities $P[N_t^0 = k]$, which are then solved using the initial conditions. He attributes the first study of the birth process to Yule (1924). The probabilities determine the transform (1). Alternatively, the transform can be obtained directly from the results in Errais et al. (2006) by observing that the birth process is an affine point process.
2. Interestingly, Cox and Ross (1976) suggest a variant of the birth process as a model for stock prices. Kou and Kou (2003) consider a birth-death process as a model for the prices of growth stocks.
3. Because it can be generated from a Poisson process, a time-changed birth process is also in the class of time-changed Lévy processes proposed by Carr and Wu (2004) to model the evolution of stock prices. To obtain a process that is self-exciting, the time change must depend on the Poisson process.
4. That is, $N - A$ is a local martingale with respect to the filtration $\mathbb{G}$.
5. A partial remedy to this feature is discussed in §4.2.5.

Acknowledgments

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