Analytical Approximations for Loan and Credit Derivatives Portfolios

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Abstract

Banks often seek to reduce the default risk exposure associated with their corporate loan portfolios by entering into credit derivative positions. They can, for example, buy default protection on selected borrowers, or diversify the portfolio by selling protection on other names. The design of suitable credit derivative positions and the estimation of risk capital reserves supporting a portfolio of loans and credit derivatives requires the computation of the probability distribution of the future portfolio value. The distribution of portfolio value takes a complicated form because it takes account of losses from default, interest income, protection and default risk premia, and the mark-to-market volatility of the derivative positions. This paper develops an analytical approximation for this distribution. The approximation is based on a small-time expansion of a transform of the portfolio value. It applies to many standard intensity-based models of firm-by-firm default timing. Numerical results illustrate the approximation for the value at risk of a portfolio of loans with hedging positions in credit swaps.

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1 Introduction

A bank making loans to corporate borrowers is exposed to losses from defaults. The exposure is often reduced by buying default protection on individual borrowers or groups of borrowers through credit or index swaps. Protection selling positions may be used to increase the degree of diversification across regions or sectors. The risk/return implications of credit derivative positions are complex, however. The swap cash flows influence the portfolio return, and the volatility of the swap values introduces additional market risk. Optimization of the risk/return profile of a portfolio of loans and credit derivatives and estimation of risk capital reserves requires calculation of the distribution of the future portfolio value under the actual probability measure. The distribution takes a complicated form because it takes account of losses from default, interest income, protection and default risk premia, and the mark-to-market volatility of the credit derivative positions. In particular, it captures the costs, benefits and risks associated with the derivative positions.

This paper develops and tests an analytical approximation for the distribution of the future value of a portfolio of loans, credit swaps and index swaps. The approximation facilitates the estimation of risk measures and capital reserves for hedged loan portfolios, the computation of margin requirements for credit swap portfolios, and other applications. Based on the contractual cash flows to the portfolio positions, we first construct a space-time Markov process governing the joint dynamics of the portfolio value, default processes, and other variables. The infinitesimal generator of this process is then used to Taylor expand a transform of the portfolio value in the horizon. Finally, we invert the expanded transform using the saddlepoint method. This approach leads to analytical approximations to the distribution of the portfolio value. Numerical results illustrate the approximation for the value at risk of a portfolio of loans with hedging positions in credit swaps.

The approximation is an alternative to Monte Carlo simulation, which may be time-consuming because the distribution of portfolio value is a nested expectation under different probability measures.[1] The inner expectation represents the mark-to-market value of the portfolio positions surviving to the horizon. The approximation is suitable for treating time-sensitive applications such as risk measure computations for short horizons. The analyticity of the approximation facilitates the estimation of sensitivities, for example the sensitivity of a risk measure to changes in the parameters of the underlying default timing model or the size of a portfolio position. Such sensitivities are often difficult to estimate using Monte Carlo. The approximation does not sacrifice the realism of Monte Carlo to a significant extent; most of the cash flows to the portfolio positions are treated exactly. The scope of the approximation is somewhat narrower than that of Monte Carlo. However, it includes most standard reduced-form models of firm-by-firm default timing.

Prior research has examined approximations for profit and loss distributions of loan and derivatives portfolios. Duffie & Pan (2001) consider a risk setting of a multi-factor jump-diffusion for default intensities and asset returns, under which between-jump returns are correlated Brownian motions, with return jumps at Poisson arrivals that are jointly normally distributed. They solve for

[1] There is a growing literature on nested simulation problems. See Ankenman, Nelson & Staum (2010), Broadie, Du & Moallemi (2011), Gordy & Juneja (2010), and Liu, Nelson & Staum (2008). The setting we face here is somewhat more complex than the ones considered in these papers because of the importance of the timing of defaults before the horizon, and the presence of cash flows generated by the swap positions before the horizon.

the characteristic function of the "delta-gamma" approximation of the portfolio value, and compute the corresponding distribution using numerical Fourier inversion. Glasserman, Heidelberger & Shahabuddin (2002) employ a related strategy in conjunction with simulation. In contrast to these articles, we expand a transform of portfolio value in the horizon, and obtain an analytical expression for the inverse of that transform using the saddlepoint method. This approach addresses important features of the portfolio value, including stochastic volatility and state-dependent jumps. Some of these features are harder to address with a delta-gamma approach.

Ait-Sahalia & Yu (2006), Figueroa-López & Houdré (2009), Rüschendorff & Woerner (2002), Ishikawa (1994) and others examine "small-time" expansions of transition densities of Markov processes governing asset prices. In the setting of our problem, there is no primitive Markov process that can be treated directly. Therefore, we first deduce an expression for the portfolio value at a fixed horizon from the contractual cash flows to the portfolio positions, then construct an appropriate space-time Markov process that comprises the portfolio value as one of its components, and finally expand the transform of that auxiliary Markov process. This approach makes explicit the change of measure from actual to risk-neutral probabilities inherent in the calculation of the distribution of future position values, and highlights the role of default risk premia.

Glasserman & Kim (2009) develop the use of saddlepoint approximations as alternatives to numerical transform inversion in the affine jump-diffusion models of Duffie, Pan & Singleton (2000). However, the value of a portfolio of loans and credit derivatives does not follow an affine jump-diffusion process, even if the relevant risk factors are assumed to be of this form. Therefore the transform of portfolio value is no longer the solution to Ricatti ODEs. We provide an expansion of this transform in terms of the infinitesimal generator of an auxiliary Markov process, and then develop saddlepoint approximations for the inverse of the expanded transform. The scope of our approach includes affine and non-affine risk-factor dynamics.

Gordy (2002) and Yang, Hurd & Zhang (2006) develop saddlepoint approximations to the distribution of loss from default in a loan portfolio. They consider static models of default timing in which the transform of portfolio loss is known explicitly. In contrast to these articles, we analyze a portfolio of loans and credit derivatives, which is exposed to loss from default and mark-to-market volatility of the derivative positions. To treat any pre-default cash flows and the mark-to-market values, we rely on dynamic reduced-form models of default timing. In our setting, the transform of the portfolio value is not known a priori but must be approximated.

Focusing on applications in portfolio optimization, Giesecke & Kim (2010) develop exact analytical expressions for the moments of the value at any horizon of a portfolio of loans, bonds and credit derivatives, assuming a doubly-stochastic model of default timing. Here we provide a short-horizon approximation for the complete distribution of the portfolio value that is valid for a much broader class of default timing models, including self-exciting models.

The rest of this paper is organized as follows. Section 2 deduces an expression for the portfolio value. Section 3 develops an expansion of a transform of portfolio value, and inverts the transform using the saddlepoint method. Section 4 illustrates the approximations and Section 5 provides numerical results. Section 6 concludes. There are several technical appendices.

2 Portfolio value

We begin by deducing a model-free expression for the value at some horizon H of a portfolio of loans, credit swaps and index swaps referenced on a collection of names $j = 1, \ldots, n$. The horizon H is taken to be shorter than the smallest maturity in the portfolio, a convention that can be relaxed at the expense of more complex notation.

A portfolio position k has notional or principal value $\delta_k \in \mathbb{R}$. For a loan, a positive δ_k indicates a loan made to an obligor. For a swap, it indicates a protection selling position. A negative δ_k indicates either a debt owed or a protection buying position. The vector $\delta = (\delta_k)$ remains fixed between time 0, when the positions are initiated, and H.

A position may generate or require intermediate cash flows, for example interest, swap premia, or recoveries. We suppose that the cash flows are invested into or funded from a money market account earning the interest r_t at time t, where r is a risk-free short rate. This is a realistic assumption as it corresponds to standard duration strategies. It also ensures consistency in both protection selling and buying positions. When there is a surplus of funds received, this convention implies a netting effect in which the payments received can be used in paying out obligations, and that payouts are otherwise funded from the money market.

To characterize the value of a position at H, suppose name j defaults at a stopping time $\tau_j > 0$, relative to a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and an information filtration $(\mathscr{F}_t)_{t\geq 0}$ satisfying the usual conditions (see, for example, Protter (2004)). Here $\mathbb{P}$ denotes the actual probability measure. The credit market is free of arbitrage opportunities, so under mild technical conditions there exists a risk-neutral pricing measure equivalent to $\mathbb{P}$. We fix a risk-neutral measure $\mathbb{P}^*$ with respect to the risk-free rate r. We let $\beta_t = \exp(-\int_0^t r_s ds)$.

2.1 Position values

Loan position. Consider a unit-principal loan maturing at $T > H$, made to a borrower j at an interest rate c_j. To ease the exposition, we suppose c_j is paid continuously through time. If j defaults before H, then the cumulative discounted payments to the lender are

$$\mathscr{D}_j^b(H) = -P_j + \int_0^H (1 - N_s^j) c_j \beta_s ds + \int_0^H (1-\ell)\beta_s dN_s^j \tag{1}$$

where P_j is the value at which the loan is issued to the borrower, $N_t^j = 1_{\{\tau_j \leq t\}}$ is the default indicator for borrower j, and ℓ is the loss rate. The first term in (1) represents the initial cash outflow, the second term the value of the interest payments, and the third term the value of the loan recovery. If the borrower survives to H, then the cumulative discounted payments to the bank are

$$\mathscr{S}_j^b(H,T) = -P_j + \int_0^H c_j \beta_s ds + \mathbb{E}^*(\mathscr{D}_j^b(T) + \beta_T(1 - N_T^j) \,|\, \mathscr{F}_H) - \mathscr{D}_j^b(H) \tag{2}$$

where $\mathbb{E}^*$ denotes expectation under the risk-neutral measure. The first term in (2) represents the initial cash outflow, the second term the value of the interest payments, and the remaining terms

the discounted time-H value of the loan, all assuming survival to H. The value of the loan at H is given by the risk-neutral expected discounted cash flows to the bank between H and T. Thus, the discounted time-H value of the loan is given by $\mathscr{P}_j^b(H,T) = N_H^j \mathscr{D}_j^b(H) + (1 - N_H^j)\mathscr{S}_j^b(H,T)$.

Credit swap position. A lender may hedge a loan made to j by buying credit swap protection on j. It may also create an exposure to j by selling protection. A protection selling position is similar to a loan, with the important difference that there is no initial cash flow required to enter into the credit swap. Consider a unit notional, protection selling position in a credit swap referenced on name j. The swap has maturity $T > H$, and is initiated at a continuously paid swap spread S_j. If j defaults before H, then the cash flows to the protection seller consist of the premiums received minus the loss paid out at default. Hence the discounted cumulative payments to the seller take the form

$$\mathscr{D}_j(H) = \int_0^H (1 - N_s^j) S_j \beta_s ds - \int_0^H \ell \beta_s dN_s^j. \tag{3}$$

If j survives to H, then the protection seller receives all premium payments scheduled during $(0, H]$; there is no default loss payment. The contract is still in force at H, and is likely to have a nonzero value due to the evolution of the market swap spread. Hence from the protection seller's point of view, the discounted cumulative cash flow takes the form

$$\mathscr{S}_j(H,T) = \int_0^H S_j \beta_s ds + \mathbb{E}^*(\mathscr{D}_j(T) \,|\, \mathscr{F}_H) - \mathscr{D}_j(H). \tag{4}$$

The last two terms in (4) represent the discounted mark-to-market value of the swap at H. This value is given by the difference between the value at H of the premium payments in $(H, T \wedge \tau_j]$ and the value at H of a potential default payment in $(H, T]$. Thus, the discounted time-H value to the protection seller of a credit swap position referenced on name j is given by $\mathscr{P}_j(H,T) = N_H^j \mathscr{D}_j(H) + (1 - N_H^j)\mathscr{S}_j(H,T)$. The corresponding value to the protection buyer, who seeks to directly hedge an exposure to j, is given by $-\mathscr{P}_j$.

Index swap position. A lender may hedge the exposure associated with a set of borrowers by buying protection through an index swap. Similarly, a lender can create a broad exposure by selling index swap protection. An index swap is referenced on a pool of $m \leq n$ credit swaps with notional $1/m$, and maturity $T > H$, which is also the maturity of the index swap. The index reference pool is a subset $\{l_1, l_2, \ldots, l_m\}$ of the set of all reference names $\{1, 2, \ldots, n\}$. The index swap protection seller covers default losses in the reference pool as they occur; the swap is not canceled at a default. The premium at time t is a fraction of the notional of the names that have survived to t. It is given by $S^x(1 - N_t/m)$, where S^x is the continuously paid index swap spread and $N^x = N^{l_1} + \cdots + N^{l_m}$ is the default count in the index portfolio. Thus $\mathscr{D}^x(H)$, the discounted stream of premiums received in $(0, H]$ less the discounted stream of default payments made during that period, is given by

$$\mathscr{D}^x(H) = \int_0^H \left(1 - \frac{N_s^x}{m}\right) S^x \beta_s ds - \frac{\ell}{m} \int_0^H \beta_s dN_s^x.$$

The mark-to-market value of the index swap at the horizon H, discounted to time 0, is

$$\mathscr{S}^x(H,T) = \mathbb{E}^*(\mathscr{D}^x(T) \,|\, \mathscr{F}_H) - \mathscr{D}^x(H)$$

The discounted time-H value of an index swap protection selling position is $\mathscr{P}^x(H,T) = \mathscr{D}^x(H) + \mathscr{S}^x(H,T)$. The value of a protection buying position is given by $-\mathscr{P}^x(H,T)$.

Loan vs. credit swap. While referencing the same borrower, a loan and a credit swap with the same maturity and notional value may be priced differently. Reasons for this may include illiquidity in the secondary loan market, the difference in the set of credit events triggering a recovery cash flow, the difference in recovery rates, the difference in capital required to enter into a position, or possible clientele trading patterns. In order to accommodate a potential valuation discrepancy between a loan and a credit swap referenced on the same borrower, we follow Chava & Jarrow (2004), Duffie, Pedersen & Singleton (2003), Liu, Longstaff & Mandell (2006) and others in making the simplistic assumption that there is an additional discount rate process φ that applies to the valuation of a loan at H. Letting $\beta_s(\varphi) = \beta_s(\varphi, H) = \beta_s \exp(-\int_H^s \varphi_u du)$ be the adjusted discount process, we then modify the survival cash flow (2) of a loan as follows:

$$\begin{aligned}\mathscr{S}_j(H,T) = &- P_j + \int_0^H c_j \beta_s ds \\ &+ \mathbb{E}^*\bigg(\int_H^T (1 - N_s^j) c_j \beta_s(\varphi) ds + \int_H^T (1-\ell)\beta_s(\varphi) dN_s^j + \beta_T(\varphi)(1 - N_T^j) \,\Big|\, \mathscr{F}_H \bigg).\end{aligned}$$

2.2 Portfolio value

The portfolio value is the weighted sum of the position values, with weights given by the notional values. Thus, the discounted time H-value is given by $V_H = \sum_k \delta_k \mathscr{P}(k, H, T_k)$, where δ_k represents the notional value of position k, $\mathscr{P}(k, H, T_k)$ is the discounted time-H value of position k, and $T_k > H$ is the maturity date of position k. Intuitively, V_H represents the total discounted net income over the horizon from premium, loan interest, and loss payments received or paid out, plus the total net mark-to-market appreciation or depreciation of the surviving positions. We have

$$\begin{aligned}V_H = \sum_{k,j} \delta_k \bigg\{ &- v_0^{jk} + a_{jk} \int_0^H (1 - N_s^j)\beta_s ds - b_{jk} \int_0^H \beta_s dN_s^j \\ &+ (1 - N_H^j)\mathbb{E}^*\bigg(a_{jk} \int_H^{T_k} (1 - N_s^j)\beta_s^k ds - b_{jk} \int_H^{T_k} \beta_s^k dN_s^j + (1 - N_T^j)\beta_T^k v_T^{jk} \,\Big|\, \mathscr{F}_H \bigg) \bigg\} \quad (5)\end{aligned}$$

where β^k is the adjusted discount process if position k represents a loan and β otherwise, v_0^{jk} is the initial loan value P_j if position k represents a loan made to j and 0 otherwise, and v_T^{jk} is the final principal repayment 1 if position k represents a loan and 0 otherwise. The coefficient a_{jk} indicates the premium relevant for position k: it is equal to c_j if k is a loan made to borrower j, S_j if k is a credit swap on j, S^x/m if k is an index swap on m names and j is in the index pool, and 0 otherwise. The coefficient b_{jk} indicates the payments at a default: it is equal to $\ell - 1$ if k is a loan made to

borrower j, ℓ if k is a credit swap on j, ℓ/m if if k is an index swap on m names and j is in the index pool, and 0 otherwise.

3 Approximating the distribution of portfolio value

The distribution of portfolio value V_H under $\mathbb{P}$ takes a complicated form: it is given by a nested expectation under different probability measures. We develop an analytical approximation to this distribution. First we Taylor expand the Laplace transform of V_H for small time horizons H in terms of the infinitesimal generator of a certain Markov process that we associate to V_H. The expanded transform is then inverted using the saddlepoint method.

3.1 Default timing

The development of the approximation requires some mild assumptions regarding default timing. Taking a standard reduced-form approach, a default indicator process N^j has intensity process λ_j under $\mathbb{P}$ satisfying $\int_0^t \lambda_j(s)ds < \infty$ almost surely. An intensity represents the conditional $\mathbb{P}$-arrival rate of a default. The evolution of λ_j under $\mathbb{P}$ is governed by the SDE

$$d\lambda_j(t) = \mu_j(t)dt + \sum_i \sigma_{j,i}(t)dW_i(t) + \sum_m dJ_m(t) + \sum_o \xi_{j,o}(t)dN_t^o \tag{6}$$

where the W_is are independent Brownian motions and the J_ms are jump processes with intensities ι_m and jump size distributions ν_m (all under $\mathbb{P}$). The intensity $\iota_m(t)$ takes the form $F_m(\Lambda(t))$ for some function F_m, where Λ is the vector of intensity processes λ_j. The coefficient $\xi_{j,o}$ represents the sensitivity of firm j's intensity to the default of firm o; $\xi_{j,j} = 0$. The function F_m, the drift μ_j, volatility $\sigma_{j,i}$ and the jump coefficients $\xi_{j,o}$ are assumed to satisfy enough regularity to guarantee a unique strong solution to the SDE (6). We do not impose any additional structure on these quantities.

By a result of Artzner & Delbaen (1995), a default indicator N^j also admits an intensity λ_j^* under the risk-neutral measure $\mathbb{P}^*$. We assume that $\lambda_j^*(t) = q(\lambda_j(t), X(t))$ for some non-random function q and vector state process X whose mth component is governed by the SDE

$$dX_m(t) = \mu_m^X(t)dt + \sum_i \sigma_{m,i}^X(t)dW_i(t), \tag{7}$$

under $\mathbb{P}$. The function q and the coefficients of (7) are chosen such that $\int_0^t \lambda_j^*(s)ds < \infty$ almost surely. This specification guarantees that the dynamics of $\lambda_j^*(t)$ under $\mathbb{P}$ are preserved in the form (6). The formulation is general enough to accommodate the relationship between the actual and risk-neutral intensities found in the empirical literature. The results of Berndt, Douglas, Duffie, Ferguson & Schranz (2008), for example, indicate a log-linear relationship.

The jump-diffusion dynamics of the intensities are fairly general. The J_m terms model jumps in an intensity between defaults, while the N^o terms model the influence of past defaults on an intensity. The W_i generate diffusive movements of an intensity between events. This formulation incorporates many of the features of default timing identified in the empirical literature, including the exposure of

firms to idiosyncratic and systematic risks, and contagion. Special cases of our formulation include the doubly-stochastic models of Duffie & Garleanu (2001), Duffie, Saita & Wang (2006), Eckner (2009), Mortensen (2006), Papageorgiou & Sircar (2007) and others, and the self-exciting models of Jarrow & Yu (2001), Giesecke, Kim & Zhu (2011) and others.

Finally, the short rate process r and the discount rate processes φ are specified as linear functions of the state process X.

3.2 Auxiliary Markov process

Consider the portfolio value process $V = (V_H)_{H<\min_k T_k}$ generated by the right side of equation (5). Clearly V is not a Markov process. We construct a suitable space-time Markov process (t, Z_t) where $Z = (Z^X, V)$ and $Z^X = (X, Y, L, \Lambda, N)$. Here, X is the vector of risk factors, $Y = \log \beta$, L is the vector of integrated discount rate processes $\int_0^{\cdot} \varphi_s ds$, Λ is the vector of intensity processes, and N is the vector of default indicators. The infinitesimal generator of the Markov process (t, Z_t) will then be used to develop an approximation to the Laplace transform of V_H.

Suppose the space-time process (t, Z_t^X) is a strong Markov process. A sufficient condition is that Z^X be a regular affine process, see Filipovic (2005). Then the process defined by

$$A_{jk}(t, T_k) = a_{jk} \int_t^{T_k} (1 - N_s^j)\beta_s^k ds - b_{jk} \int_t^{T_k} \beta_s^k dN_s^j + (1 - N_{T_k}^j)\beta_{T_k}^k v_{T_k}^{jk} \tag{8}$$

for $t < T_k$ is also a strong Markov process relative to the filtration generated by (t, Z_t^X), by Theorem V.35 of Protter (2004). Thus, there is a real-valued function $f^{jk}(t, z)$ satisfying

$$f^{jk}(t, Z_t) = \mathbb{E}^*(A_{jk}(t, T_k) \,|\, \mathscr{F}_t) \tag{9}$$

on the no-default set $\{N_t^j = 0\}$, for $t < H$. The conditional expectation in (9) is a function of the risk-neutral intensity $\lambda_j^*(t)$, which is, in turn, a function of the actual intensity $\lambda_j(t)$ and X_t. Hence the arguments of $f^{j,k}$ can be expressed in terms of the actual intensity instead of the risk-neutral intensity. The function $f^{jk}(t, Z_t)$ depends on Z_t through $(X, Y, L, \Lambda)_t$ but not N_t or V_t. By Itô's formula, there exist functions μ_{jk}^f and σ_{jki}^f such that

$$\begin{aligned} f^{jk}(t, Z_t) &= f^{jk}(0, Z_0) + \int_0^t \mu_{jk}^f(s, Z_s) ds + \sum_i \int_0^t \sigma_{jki}^f(s, Z_s) dW_i(s) \\ &+ \sum_{0<s\le t} \left[f^{jk}(s, Z_s) - f^{jk}(s, Z_{s-}) \right]. \end{aligned} \tag{10}$$

Now return to the portfolio value process $V = (V_H)_{H<T}$ generated by the right side of equation

(5) for $T = \min_k T_k$. By (9), for $t < T$, we have

$$\begin{aligned} V_t &= \sum_{k,j} \delta_k \Big\{ -v_0^{jk} + a_{jk} \int_0^t (1 - N_s^j) e^{-Y_s} ds \\ &\qquad -b_{jk} \int_0^t e^{-Y_s} dN_s^j + (1 - N_t^j) f^{jk}(t, Z_t) \Big\}. \end{aligned} \tag{11}$$

By Itô's formula and (10), for $t < T$ we obtain

$$\begin{aligned} V_t &= V_0 + \sum_{k,j} \delta_k \Big\{ \int_0^t (1 - N_s^j)(a_{jk} e^{-Y_s} + \mu_{jk}^f(s, Z_s)) ds \\ &\qquad + \int_0^t (1 - N_s^j) \sum_i \sigma_{jki}^f(s, Z_s) dW_i(s) + \sum_{0<s\le t} \Delta V_s \Big\}. \end{aligned} \tag{12}$$

The SDE (12) determines the final v-component $\mu^v(t, Z_t)$ of the drift vector $\mu(t, Z_t)$ and the v-th row $\sigma^v(t, Z_t)$ of the diffusion matrix $\sigma(t, Z_t)$ of the vector process $(t, Z_t) = (t, X_t, Y_t, L_t, \Lambda_t, N_t, V_t)$. The other components and rows follow directly from (6) and (7). We have

$$\mu^v(t, Z_t) = \sum_{k,j} \delta_k (1 - N_t^j)(a_{jk} e^{-Y_t} + \mu_{jk}^f(t, Z_t)) \tag{13}$$

$$\sigma^v(t, Z_t) = \Big(\sum_{k,j} \delta_k (1 - N_t^j) \sigma_{jki}^f(t, Z_t) \Big)_i. \tag{14}$$

Below we will show that, if (t, Z^X) is a strong Markov process, then so is (t, Z_t). Under this assumption, we deduce the infinitesimal generator of the Markov process (t, Z_t). The infinitesimal generator is the operator $\mathscr{A}$ such that $\{h(t, Z_t) - \int_0^t \mathscr{A} h(s, Z_s) ds : t \ge 0\}$ is a martingale for all bounded C^2 functions h with bounded first and second derivatives.[2] By Itô's formula,

$$\begin{aligned} h(t, Z_t) &= h(0, Z_0) + \int_0^t (h_t(s, Z_s) + h_z(s, Z_s)\mu(s, Z_s) + \frac{1}{2} \mathrm{tr}\,[h_{zz}(s, Z_s)\sigma(s, Z_s)\sigma(s, Z_s)^\top]) ds \\ &+ \int_0^t h_z(s, Z_s)\sigma(s, Z_s) dW(s) + \sum_{0<s\le t} [h(s, Z_s) - h(s, Z_{s-})] \end{aligned} \tag{15}$$

where W is the vector of the independent Brownian motions W_i. Under sufficient regularity, the integral against W is a martingale. Formula (15) indicates that in order to obtain a martingale, we must subtract the drift and the compensator of the jumps of $h(t, Z_t)$ from $h(t, Z_t)$. It is important to note that while the probability of simultaneous jumps across independent pure jump processes is zero, we do need to account for common jumps in the components because both Λ and V are driven by the various jump processes. For instance, if N^j jumps this causes a jump in the Λ components of Z via the feedback terms, and V by causing the default payment and knocking-off the mark-to-market.

[2] The second order conditions apply only to the space derivatives. In the time variable h only needs to be continuously differentiable with a bounded first derivative.

The jump-term can be rewritten in terms of the individual contributions of each pure jump process N^j and J_m, but the summand must keep track of these synchronous jumps.

The compensator of the jumps of $h(s, Z_s)$ in $[0, t]$ induced by jumps in the N^j is given by

$$\int_0^t \sum_j \lambda_j(Z_{s-})(1 - N^j_{s-}) \left[h(s, Z_{s-} + w_j(s, Z_{s-})) - h(s, Z_{s-})\right] ds$$

where $w_j = w_j(s, Z_{s-})$ is the appropriate jump size vector, given in Appendix B. Similarly, the compensator of the jumps of $h(s, Z_s)$ in $[0, t]$ induced by jumps in the J_m is given by

$$\int_0^t \int \sum_m \iota_m(Z_{s-}) \left[h(s, Z_{s-} + \varsigma_m(s, Z_{s-}, \epsilon)) - h(s, Z_{s-})\right] d\nu_m(\epsilon) ds$$

where $\varsigma_m(\epsilon) = \varsigma_m(s, Z_{s-}, \epsilon)$ is the appropriate jump size vector, given in Appendix B. Under sufficient regularity conditions, which are once again assumed, the compensated jumps of $h(s, Z_s)$ in $[0, t]$ are martingales. Then the generator $\mathscr{A}$ of (t, Z_t) is given by

$$\begin{aligned}
\mathscr{A}h(t, z) &= h_t(t, z) + h_z(t, z)\mu(t, z) + \frac{1}{2}\mathrm{tr}\left[h_{zz}(t, z)\sigma(t, z)\sigma(t, z)^\top\right] \\
&+ \sum_m \iota_m(z) \int \left[h(t, z + \varsigma_m(t, z, \epsilon)) - h(t, z)\right] d\nu_m(\epsilon) \\
&+ \sum_j \lambda_j(z)(1 - n_j) \left[h(t, z + w_j(t, z)) - h(t, z)\right].
\end{aligned}$$

Next, we use the generator $\mathscr{A}$ to approximate the Laplace transform of V_H.

3.3 Expanding the transform

The definition of the generator $\mathscr{A}$ implies that for any horizon $\Delta > 0$,

$$\mathbb{E}(h(\Delta, Z_\Delta)|Z_0 = z) = h(0, z) + \mathbb{E}\bigg(\int_0^\Delta \mathscr{A}h(s, Z_s) ds \;\bigg|\; Z_0 = z \bigg).$$

For "small" Δ, the conditional expectation can be approximated to first order by $\Delta\mathscr{A}h(0, z)$. A more accurate approximation is obtained by including the contributions of higher-order terms. For a C^{M+2} function h, the corresponding "small-time" expansion is given as

$$\mathbb{E}(h(\Delta, Z_\Delta)|Z_0 = z) = \sum_{m=0}^{M} \frac{\Delta^m}{m!} \mathscr{A}^m h(0, z) + O(\Delta^{M+1}). \tag{16}$$

By applying the Taylor series (16) in Δ around their continuous-time limit to the function $h(t, z) = e^{uz_v}$, where u is a real number and z_v is the v-component of the vector z, we obtain a small-time

approximation to the Laplace transform of V_Δ as

$$\psi(\Delta, u|z) = \mathbb{E}(e^{uV_\Delta}|Z_0 = z) \approx \sum_{m=0}^{M} \frac{\Delta^m}{m!} \mathscr{A}^m e^{uz_v} \tag{17}$$

for those $u \in \mathbb{R}$ for which $\psi(\Delta, u|z) < \infty$. For this expansion to be legitimate, we need to verify that the space-time process (t, Z_t) is a well-defined strong Markov process, and that e^{uz_v} is in the domain of the generator $\mathscr{A}$. A sufficient condition for this is given below.

Proposition 3.1. *Suppose that (t, Z_t^X), where $Z^X = (X, Y, L, \Lambda, N)$, is a strong Markov process. Then the augmented process (t, Z_t), where $Z = (X, Y, L, \Lambda, N, V)$, is also a strong Markov process and e^{uz_v} is in the domain $D_{\mathscr{A}}$ of its generator $\mathscr{A}$.*

The proof is found in Appendix A. While the conclusion holds for a broader class of processes, regular affine processes as defined by Duffie, Filipovic & Schachermayer (2003) have a sufficiently wide scope for our application.

To recover the distribution of V_Δ from its transform, it is convenient to work with the cumulant generating function (CGF) of V_Δ, defined by $K(\Delta, u|z) = \log \psi(\Delta, u|z)$. Let $K^{(n)}$ be the n-th order approximation of K. We have

$$K^{(n)}(\Delta, u|z) = uz_v + \frac{\partial}{\partial \Delta} K(\Delta, u|z)\Big|_{\Delta=0} \Delta + \frac{1}{2} \frac{\partial^2}{\partial \Delta^2} K(\Delta, u|z)\Big|_{\Delta=0} \Delta^2 + \cdots + O(\Delta^{n+1}).$$

Following (16), these expressions can be found explicitly by performing the n-th order Taylor expansion on $\log \sum_{m=0}^{n} \frac{\Delta^m}{m!} \mathscr{A}^m e^{uz_v}$. The first and second order approximations are given below. Higher-order terms can be developed similarly. To simplify the notation, we suppress the (t, x, y, l, n) dependence of the coefficient functions.

Proposition 3.2. *The first and second order approximations to the CGF of V_Δ are given by*

$$\begin{aligned} K^{(1)}(\Delta, u|z) &= uz_v + c_1 \Delta \\ K^{(2)}(\Delta, u|z) &= uz_v + c_1 \Delta + \left(-\frac{c_1^2}{2} + c_2 \right) \Delta^2 \end{aligned}$$

where formulas for $c_1 = \frac{\mathscr{A} e^{uz_v}}{e^{uz_v}}$ and $c_2 = \frac{1}{2} \frac{\mathscr{A}^2 e^{uz_v}}{e^{uz_v}}$ are given in Appendix C.

3.4 Recovering the distribution

We obtain an approximation to the distribution of V_Δ by inverting the approximate transform (17) using the saddlepoint method. An advantage of the saddlepoint method over alternative numerical inversion approaches is that it leads to analytical expressions for the approximate distribution of V_Δ. The saddlepoint method expands the inversion integral

$$p(\Delta, y|z) = \frac{1}{2\pi \mathbf{i}} \int_{\hat{u} - \mathbf{i}\infty}^{\hat{u} + \mathbf{i}\infty} \exp(-uy) \psi(\Delta, u|z) du \tag{18}$$

which represents the density of V_Δ at $y \in \mathbb{R}$ provided that $\psi(\Delta, \cdot\,|z) \in L^1(\mathbb{R})$. Here, $\hat{u}$ is a real value so that the contour path of integration is in the region of convergence of $\psi(\Delta, u|z)$. The method was proposed by Daniels (1954), and relies on a certain choice of $\hat{u}$. Under mild regularity conditions, the CGF $K(\Delta, u|z)$ is strictly convex in u. Consider the value $\hat{u} \in \mathbb{R}$, called the saddlepoint, given by the solution in u to the equation

$$\frac{\partial K(\Delta, u|x)}{\partial u} = y.$$

The modulus of the integrand in the contour integral (18) is minimized along the real axis at $\hat{u}$, and maximized at $\hat{u}$ along the contour parallel to the imaginary axis passing through $\hat{u}$. Hence the region outside a neighborhood of the saddlepoint contributes little to the integration, and the density can be approximated by Taylor expanding the exponent $K(\Delta, u|x) - uy$ around $\hat{u}$.

Using different leading terms in the approximation yields different saddlepoint formulas. As in Wood, Booth & Butler (1993), these formulas can be found by applying a change of variables argument to the original Gaussian leading term expansions of Daniels (1954) or Lugannani & Rice (1980). In our setting, because of the presence of jumps, we propose to use a non-Gaussian base distribution that corresponds to the CGF $K_0(\Delta, u|z)$ satisfying

$$\begin{aligned} e^{K_0(u|z)-uz_v} &= \left(1 - \Delta \sum_j \lambda_j(z)(1-n_j)\right) \exp\left(\mu^v(\Delta, z)\Delta u + \frac{1}{2}\left[\sigma(\Delta, z)\sigma(\Delta, z)^\top\right]^{vv} \Delta u^2\right) \\ &+ \Delta \sum_m \iota_m(z) \int_{\mathbb{R}} e^{-u\sum_k \delta_k F^{j,k}(\Delta, z, \epsilon)} d\nu_m(\epsilon) + \Delta \sum_j \lambda_j(z)(1-n_j) e^{-u\sum_k \delta_k G^{j,k}(\Delta, z)} \end{aligned}$$

where $F^{j,k}(\Delta, z, \epsilon)$, and $G^{j,k}(\Delta, z)$ represent the jump sizes of the (j,k) cross-section of z_v caused by the J_m and N_j jumps, respectively. See Appendix B for details. This is the distribution where a single jump is permitted during the horizon (see Ait-Sahalia & Yu (2006) for a discussion). Letting $\Phi(\cdot; m, v)$ denote the normal distribution function with mean m and variance v, the corresponding distribution function $F_0(a|z)$, $a \in \mathbb{R}$, takes the form

$$\begin{aligned} F_0(a|z) &= \left(1 - \Delta \sum_j \lambda_j(z)(1-n_j) - \Delta \sum_m \iota_m(z)\right) \Phi\left(a; z_v + \mu^v(\Delta, z)\Delta, \left[\sigma(\Delta, z)\sigma(\Delta, z)^\top\right]^{vv} \Delta\right) \\ &+ \Delta \sum_j \lambda_j(z)(1-n_j) \mathbf{1}_{\{a = z_v + \sum_k \delta_k G^{jk}(\Delta, z)\}} + \Delta \sum_m \iota_m(z) \nu_m\left(a - z_v\right). \end{aligned} \tag{19}$$

With the true CGF K replaced by its nth order approximation $K^{(n)}$ and the non-Gaussian base distribution defined above, we obtain our approximation of the cumulative distribution function of V_Δ. Let $n_1 \geq 0$ and $n_2 \geq 1$ be the orders of the saddlepoint and CGF approximations, respectively. The $n_1 = 0, n_2 = n$ order approximation is given by

$$\mathbb{P}(V_\Delta \leq a | Z_0 = z) \approx F_0(a|z) - f_0(a|z) \left\{ \frac{1}{\hat{u}} (K_0''(\hat{w}|z))^{1/2} \left(\frac{\partial^2 K^{(n)}(\Delta, \hat{u}|z)}{\partial u^2} \right)^{-1/2} - \frac{1}{\hat{w}} \right\} \tag{20}$$

where $F_0(a|z)$ is given by formula (19) and $f_0(a|z)$ is the associated density function. Higher order

(in n_1) terms can be computed by modifying the higher-order Lugannani & Rice (1980) formulas appropriately. The saddlepoints $\hat{u}$ and $\hat{w}$ satisfying

$$\begin{aligned} a &= \frac{\partial}{\partial u} K^{(n)}(\Delta, \hat{u}|z), \\ a &= \frac{\partial}{\partial u} K_0(\hat{w}|z) \end{aligned}$$

can be computed numerically or by the series reversion method of Lieberman (1994).

3.5 Order of approximation errors

The overall approximation is a 2-stage process. A modification of Theorem 3 of Ait-Sahalia & Yu (2006) provides us with a characterization of the compound Δ-order of the approximation. If $n_1 \geq 0$ and $n_2 \geq 1$ are the orders of the saddlepoint and CGF approximations, respectively, then

$$p(\Delta, y|x) = p^{(n_1,n_2)}(\Delta, y|x) \left(1 + O\left(\Delta^{\min(n_1+1, n_2/2)}\right)\right), \tag{21}$$

where $p^{(n_1,n_2)}(\Delta, y|x)$ is the approximate density. For typical saddlepoint approximations of order $n_1 = 0, 1$, the precision of the CGF approximation will be the dominant factor.

4 Examples

We illustrate the approximation in standard model formulations, obtained by specializing into the default intensity and risk factor dynamics of Section 3.1.

4.1 Doubly-stochastic model

Our first example is a doubly-stochastic model of firm intensities, as in Berndt et al. (2008), Chava & Jarrow (2004), Duffie & Garleanu (2001), Duffie et al. (2006), Eckner (2009), Feldhütter & Nielsen (2010), Mortensen (2006), Papageorgiou & Sircar (2007) and many others. We must specify the actual and risk-neutral intensities.

Actual intensity. The risk factors X_m follow $\mathbb{P}$-Feller diffusions

$$dX_m(t) = \kappa_{X_m}[\theta_{X_m} - X_m(t)]dt + \sigma_{X_m}\sqrt{X_m(t)}dW_m(t), \qquad m = 1, 2, \ldots, n+K \tag{22}$$

where $2\kappa_{X_m}\theta_{X_m} > \sigma^2_{X_m}$ so that the processes remain strictly positive almost surely. (We always assume this condition when a new Feller diffusion is introduced.) The actual intensities are

$$\lambda_j(t) = X_j(t) + \sum_{k=1}^{K} \omega_j^k X_{n+k}(t), \qquad j = 1, 2, \ldots, n \tag{23}$$

where ω_j^k is the factor loading of the kth common risk factor for firm j. The ω_j^ks control and differentiate the level of exposure that a firm has to the common risk factors, and therefore they modulate the default correlation structure. As shown by Duffie & Garleanu (2001), certain restrictions apply to the parameters for the resulting firm intensity process to remain a $\mathbb{P}$-Feller diffusion. Denoting the parameter triplets of the risk factors by $(\kappa_{X_m}, \theta_{X_m}, \sigma_{X_m})$, we require that

$$\kappa_{X_m} = \kappa, \quad m = 1, \ldots, n+K \tag{24}$$

$$\sigma_{X_j} = \sqrt{\omega_j^k}\sigma_{X_{n+k}}, \quad k = 1, \ldots, K, \text{ for each } j = 1, \ldots, n. \tag{25}$$

Conditional on a realization of $(X_j, X_{n+1}, \ldots, X_{n+K})$, firm j defaults at the first jump time of an inhomogeneous Poisson process with (conditionally deterministic) intensity λ_j. Given a realization of the systematic factors $X_{n+1}, \ldots, X_{n+K}$, which drive the intensities of all firms, the default times are $\mathbb{P}$-independent of one another. Our parametrization of the risk premia, described next, preserves this property under the risk-neutral measure.

Risk-neutral intensity. Berndt et al. (2008) provide regression results and an empirical time-series fitting in which the risk-neutral intensity is modeled as an affine function of the actual intensity plus a stochastic noise term. Motivated by this formulation, we model the relationship between the actual and risk-neutral intensities as

$$\lambda_j^*(t) = \hat{\alpha}_j + \gamma_j \lambda_j(t) + \hat{u}_j(t), \qquad j = 1, 2, \ldots, n \tag{26}$$

where $\hat{\alpha}_j$ and γ_j are parameters and the $\hat{u}_j$s are stochastic noise terms. With (23),

$$\lambda_j^*(t) = \hat{\alpha}_j + \gamma_j X_j(t) + \gamma_j \sum_{k=1}^{K} \omega_j^k X_{n+k}(t) + \hat{u}_j(t). \tag{27}$$

Since $X_j, X_{n+1}, \ldots, X_{n+K}$ are $\mathbb{P}$-Feller diffusions, λ_j^* will also follow a Feller diffusion under $\mathbb{P}$ as long as the noise term $\hat{u}_j$ is a $\mathbb{P}$-Feller diffusion, and parameter restrictions are met.

In the $\mathbb{P}$-dynamics of λ_j^* given by (27), the λ_j^*s remain correlated through the common risk factors $X_{n+1}, \ldots, X_{n+K}$. Given the structure of the transformation (27), we will decompose (26) into $K+1$ independent transformations – a component attributed to the transformation of the idiosyncratic factor process $X_j \to X_j^*$, and K others describing $X_{n+k} \to X_{n+k}^*$, $k = 1, \ldots, K$. We achieve (26) by an explicit specification of these transformations. We assume that there are $2n+2K$ state variables – n idiosyncratic risk factors, K common factors, and $n+K$ factors that modulate the relationship between the actual and risk-neutral factors. The modulator terms, denoted by u_m, are $\mathbb{P}$-Feller diffusions that track the stochastic evolution of the default timing risk premia. We posit the following relationship between the factors X_m and their risk-neutral counterparts X_m^*:

$$X_m^*(t) = \alpha_m + \gamma_m X_m(t) + u_m(t), \qquad m = 1, 2, \ldots, n+K \tag{28}$$

with $\gamma_{n+1} = \ldots = \gamma_{n+K} = 1$ for the common factors. Here the α_ms and γ_ms are parameters and the

u_ms are again independent $\mathbb{P}$-Feller diffusions with dynamics given by

$$du_m(t) = \kappa_{u_m}[\theta_{u_m} - u_m(t)]dt + \sigma_{u_m}\sqrt{u_m(t)}dB_m(t), \qquad m = 1, 2, \ldots, n+K. \tag{29}$$

The B_ms are independent Brownian motions under $\mathbb{P}$, also independent of the W_ms of (22). By (27),

$$\lambda_j^*(t) = \alpha_j + \gamma_j X_j(t) + u_j(t) + \gamma_j \sum_{k=1}^{K} \omega_j^k(\alpha_{n+k} + X_{n+k} + u_{n+k}(t)) \tag{30}$$

$$= X_j^*(t) + \gamma_j \sum_{k=1}^{K} \omega_j^k X_{n+k}^*(t) \tag{31}$$

which gives us a factor decomposition of the risk-neutral intensity λ_j^*, analogous to (23) of the actual intensity λ_j. Just as $(X_j, X_{n+1}, \ldots, X_{n+K})$ drives λ_j, the transformed factors $(X_j^*, X_{n+1}^*, \ldots, X_{n+K}^*)$ now drive λ_j^*. Equation (30) gives us our intended formulation of (26) where $\hat{\alpha}_j = \alpha_j + \gamma_j \sum_{k=1}^{K} \omega_j^k \alpha_{n+k}$, and $\hat{u}_j = u_j + \gamma_j \sum_{k=1}^{K} \omega_j^k u_{n+k}$. There is a correlation structure that is also embedded in the stochastic noise term $\hat{u}_j$, which captures fluctuations in the default timing risk premium. The process $\hat{u}_j$ is a $\mathbb{P}$-Feller diffusion under the parameter restrictions

$$\kappa_{u_j} = \kappa, \quad j = 1, \ldots, n+K \tag{32}$$

$$\sigma_{u_j} = \sqrt{\omega_j^k \gamma_j}\sigma_{X_{n+k}}, \quad k = 1, \ldots, K, \text{ for each } j = 1, \ldots, n+K \tag{33}$$

with $\omega_j^k = \gamma_j = 1$ for $j > n$. These restrictions ensure that the risk-neutral factors and intensities remain $\mathbb{P}$-Feller diffusions.

We have specified the $\mathbb{P}$-dynamics of λ_j^*. In order to obtain the $\mathbb{P}^*$-dynamics of λ_i^*, which are required for the mark-to-market valuations, we only need to specify how the Brownian motions driving $(X_1, \ldots, X_{n+K}, u_1, \ldots, u_{n+K})$ change under $\mathbb{P}^*$. Let η be a fixed parameter. Given the actual measure $\mathbb{P}$, we define the risk-neutral measure $\mathbb{P}^*$ to be the measure under which, along with the previous intensity transformations for the default point processes, $(W_1^*, \ldots, W_{n+K}^*, B_1^*, \ldots, B_{n+K}^*)$ is a $(2n+2K)$-dimensional standard Brownian motion where

$$W_m(t) = W_m^*(t) - \int_0^t \eta \left(\frac{X_m(s)}{\sigma_{X_m}}\right)^{1/2} ds, \qquad m = 1, 2, \ldots, n+K \tag{34}$$

$$B_m(t) = B_m^*(t) - \int_0^t \eta \left(\frac{u_m(s)}{\gamma_m \sigma_{X_m}}\right)^{1/2} ds, \qquad m = 1, 2, \ldots, n+K. \tag{35}$$

The parameter η governs the risk premium for diffusive volatility of mark-to-market values. It is appropriately scaled to adjust the Brownian motions driving the individual factors and modulators. This synchronization preserves model parsimony and enforces the κ restrictions that ensure that the overall risk-neutral intensity processes remain $\mathbb{P}^*$-Feller diffusions.

The overall measure change that governs both the point process and diffusion transformations can be made precise by the corresponding Radon-Nikodym derivative, which is a product of the exponen-

tial martingale terms for the independent transforms. Our default arrival process, by construction, remains doubly-stochastic in the risk-neutral measure.

Short rate and discount rate processes. We preserve the structural coherence of our framework and subsequent computations by modeling the short rate and discount rate processes as multi-factor Feller diffusions as well. We let

$$r = R(X) = \sum_{m \in I_R} \omega_m^r X_m,$$
$$\varphi^k = L^k(X) = \sum_{m \in I_{L_k}} \omega_m^{l_k} X_m,$$

where I_R and I_{L_k} are subsets of the $\mathbb{P}$-Feller risk factors. Some may overlap with the common factors introduced previously, while others are idiosyncratic. This formulation allows both processes to remain almost surely positive, and permits the flexibility of cross correlations between the intensities, short rate, and discount rates. Factors can be excluded by setting the corresponding loadings to 0.

Approximation. To simplify the notation, we take X to encompass all idiosyncratic and common risk-factors as well as the measure-change and discount rate factors. Elements indexed by $1, 2, \ldots, n$ correspond to the idiosyncratic factors, $n+1, \ldots, n+p$ the common factors, $n+p+1, \ldots, 2n+2p$ the measure change factors, and $2n+2p+1, \ldots, 2n+2p+q$ the discount rate factors. The portfolio value process V_t is the same as (5), but now that the intensities are deterministic functions of X_t, the mark-to-market portion of the portfolio value, $f^{jk}(t, Z_t)$, becomes effectively a function of t, X_t, Y_t and L_t only. On $\{N_t^j = 0\}$, by the doubly stochastic property,

$$\begin{aligned} f^{jk}(t, Z_t) &= a_{jk} \int_t^T \mathbb{E}^*\left[\beta_v^k \,|\, \mathscr{F}_t\right] \mathbb{E}^*\left[e^{-\int_t^v \lambda_j^*(s)ds} \,|\, \mathscr{F}_t\right] dv \\ &- b_{jk} \int_t^T \mathbb{E}^*\left[r_v^k \beta_v^k \,|\, \mathscr{F}_t\right] \mathbb{E}^*\left[1 - e^{-\int_t^v \lambda_j^*(s)ds} \,|\, \mathscr{F}_t\right] dv \\ &+ (v_T^{jk} + b_{jk}) \mathbb{E}^*\left[\beta_T^k \,|\, \mathscr{F}_t\right] \mathbb{E}^*\left[e^{-\int_t^T \lambda_j^*(s)ds} \,|\, \mathscr{F}_t\right] - b_{jk} \mathbb{E}^*\left[\beta_T^k \,|\, \mathscr{F}_t\right] \end{aligned}$$

for the adjusted interest rate process $r^k = r + \varphi^k$ and $\beta^k = e^{-Y - L^k}$. We can compute $f^{jk}(t, Z_t)$ and its various derivatives explicitly via extended transforms of Feller diffusions, and the affine measure change parametrization. This heavily relies on the affine jump-diffusion results of Duffie et al. (2000). See Appendix D for details. Given the factor dynamics,

$$df^{jk}(t, Z_t) = \mu_{jk}^f(t, Z_t)dt + \sum_i \sigma_{X_i} \sqrt{X_i(t)} f_{x_i}^{jk}(t, Z_t) dW_i(t)$$

where the drift $\mu_{jk}^f(t, Z_t)$ is given by

$$f_t^{jk}(t, Z_t) + R(X_t) + \sum_m L^m(X_t) + \sum_i \left\{ \kappa(\theta_{X_i} - X_i(t)) f_{x_i}^{jk}(t, Z_t) + \frac{1}{2} \sigma_{X_i}^2 X_i(t) f_{x_i x_i}^{jk}(t, Z_t) \right\}.$$

Let $g^{jk}(t, Z_t) = b_{jk}e^{-Y_t} + f^{jk}(t, Z_t)$. Moreover,

$$\begin{aligned} dV_t &= \sum_{k,j} \delta_k (1 - N_t^j) \Big\{ \Big(a_{jk} e^{-Y_t} + \mu_{jk}^f(t, Z_t) \Big) \, dt + \sum_i \sigma_{X_i} \sqrt{X_i(t)} f_{x_i}^{jk}(t, Z_t) dW_i(t) \Big\} \\ &- \sum_{k,j} \delta_k g^{jk}(t, Z_{t-}) dN_t^j. \end{aligned} \tag{36}$$

Following the general method, we look at the augmented vector Markov process (t, Z_t). The individual components of the drift $\mu(t, Z_t)$ can be expressed as

$$\begin{aligned} \mu^{x_i}(t, Z_t) &= \kappa(\theta_{X_i} - X_i(t)) \\ \mu^{y}(t, Z_t) &= R(X_t) \\ \mu^{l_k}(t, Z_t) &= L^k(X_t) \\ \mu^{n_i}(t, Z_t) &= 0 \\ \mu^{v}(t, Z_t) &= \sum_{k,j} \delta_k (1 - N_t^j) \left(a_{jk} e^{-Y_t} + \mu_{jk}^f(t, Z_t) \right). \end{aligned}$$

Similarly, the rows of the diffusive volatility matrix $\sigma(t, Z_t)$ are given as

$$\begin{aligned} \sigma^{x_i}(t, Z_t) &= \mathbf{e_i} \cdot \sigma_{X_i} \sqrt{X_i(t)} \\ \sigma^{y}(t, Z_t) &= \sigma^{l_k}(t, Z_t) = \sigma^{n_i}(t, Z_t) = \mathbf{0} \\ \sigma^{v}(t, Z_t) &= \left(\sum_{k,j} \delta_k (1 - N_t^j) \sigma_{X_i} \sqrt{X_i(t)} f_{x_i}^{jk} \right)_i. \end{aligned}$$

The X, Y and L^k components do not have jumps while the N components of Z have unit jumps. The final element V in the vector Z has jumps of size $-\sum_k \delta_k g^{jk}(t, Z_t)$ for each jump in N_t^j. Since the jumps in V only come from jumps in N, the jump sizes are completely determined by (t, X_t, Y_t, L_t). This simplifies the expansions. The expressions for c_1 and c_2 in Proposition 3.2 simplify to

$$c_1 = \mu^v u + \frac{1}{2} \left[\sigma \sigma^\top \right]^{vv} u^2 + \sum_j \lambda_j(z)(1 - n_j) \left(\exp\left(-u \sum_k \delta_k g^{jk} \right) - 1 \right),$$

where we suppress the arguments of the functions μ, σ and $g^{j,k}$, and

$$\begin{aligned} 2c_2 &= \mu_t^v u + \frac{1}{2} \left[\sigma \sigma^\top \right]_t^{vv} u^2 - \sum_j \lambda_j(z)(1 - n_j) u \sum_k \delta_k g_t^{jk} e^{-u \sum_m \delta_m g^{j,m}} \\ &+ \left\{ \mu_z^v u + \frac{1}{2} \left[\sigma \sigma^\top \right]_z^{vv} u^2 + \left[\sum_j \lambda_j(z)(1 - n_j) \left(e^{-u \sum_k \delta_k g^{jk}} - 1 \right) \right]_z \right\} \mu \\ &+ \left\{ \mu^v u^2 + \frac{1}{2} \left[\sigma \sigma^\top \right]^{vv} u^3 + u \sum_j \lambda_j(z)(1 - n_j) \left(e^{-u \sum_k \delta_k g^{jk}} - 1 \right) \right\} \mu^v \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}\mathrm{tr}\left[\left\{\mu^v_{zz}u + \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv}_{zz}u^2 + \left[\sum_j \lambda_j(z)(1-n_j)\left(e^{-u\sum_k \delta_k g^{jk}} - 1\right)\right]_{zz}\right\}\sigma\sigma^\top\right] \\
&+ \quad \left\{\mu^v_z u^2 + \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv}_{z}u^3 + u\left[\sum_j \lambda_j(z)(1-n_j)\left(e^{-u\sum_k \delta_k g^{jk}} - 1\right)\right]_{z}\right\}\left[\sigma\sigma^\top\right]^{\cdot z} \\
&+ \quad \frac{1}{2}\left\{\mu^v u^3 + \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv}u^4 + u^2\sum_j \lambda_j(z)(1-n_j)\left(e^{-u\sum_k \delta_k g^{jk}} - 1\right)\right\}\left[\sigma\sigma^\top\right]^{vv} \\
&+ \quad \sum_j \lambda_j(z)(1-n_j)\left\{\left(\mu^v_{w_j+}u + \frac{1}{2}\left[\sigma_{w_j+}\sigma^\top_{w_j+}\right]^{vv}u^2 + \left[\sum_l \lambda_l(x)\left(e^{-u\sum_k \delta_k g^{l,k}} - 1\right)\right]_{w_j+}\right)\right. \\
&\qquad\qquad \left.\times e^{-u\sum_k \delta_k g^{jk}} - \mu^v u - \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv}u^2 - \sum_l \lambda_l(x)(1-n_l)\left(e^{-u\sum_k \delta_k g^{l,k}} - 1\right)\right\}
\end{aligned}
$$

where $\mu^v_{j+} = \mu^v(t,x,y,n+e_j)$ and $\left[\sigma_{j+}\sigma^\top_{j+}\right]^{vv} = \left[\sigma(t,x,y,n+e_j)\sigma(t,x,y,n+e_j)^\top\right]^{vv}$.

4.2 Self-exciting model

In the doubly-stochastic model examined previously, defaults are correlated because firms are exposed to a common risk factor. We next illustrate the approximation on a self-exciting formulation, in which defaults are correlated because a default has an impact on the surviving firms, as in Jarrow & Yu (2001), Errais, Giesecke & Goldberg (2010) and others.

Actual intensity. The actual intensity follows

$$
d\lambda_j(t) = \kappa_j(\theta_j - \lambda_j(t))dt + \sum_o \xi_{j,o}dN^o_t \tag{37}
$$

where κ_j and θ_j are the reversion speed and level parameters, respectively, and $\xi_{j,o}$ models the impact of firm o's default on the intensity of firm j.

Risk-neutral intensity. For risk-premium parameters α_j and β_j, the risk-neutral intensity follows

$$
\lambda^*_j(t) = \alpha_j + \beta_j\lambda_j(t). \tag{38}
$$

Short rate and discount rate processes. We assume the same model as the previous section for the diffusive short rate factors, and the corresponding diffusive volatility risk premia.

Approximation. Given the factor and intensity dynamics, we have

$$
\begin{aligned}
df^{jk}(t,Z_t) &= \mu^f_{jk}(t,Z_t)dt + \sum_i \sigma_{X_i}\sqrt{X_i(t)}f^{jk}_{x_i}(t,Z_t)dW_i(t) \\
&+ \sum_o \left[f^{jk}(t,Z_{t-} + \xi_{j,o}\mathbf{e}_{\lambda_\mathbf{j}}) - f^{jk}(t,Z_{t-})\right]dN^o_t
\end{aligned}
$$

where

$$\begin{aligned}\mu^f_{jk}(t,Z_t) &= f^{jk}_t(t,Z_t) + R(X_t) + \sum_m L^m(X_t) + \sum_j \kappa_j(\theta_j - \lambda_j(t)) f^{jk}_{\lambda_j} \\ &+ \sum_i \left\{ \kappa_{X_i}(\theta_{X_i} - X_i(t)) f^{jk}_{x_i}(t,Z_t) + \frac{1}{2}\sigma^2_{X_i} X_i(t) f^{jk}_{x_i x_i}(t,Z_t) \right\}.\end{aligned}$$

Let $g^{jk}(t,Z_t) = b_{jk}e^{-Y_t} + f^{jk}(t,Z_t)$. The dynamics of the portfolio value are

$$\begin{aligned}dV_t &= \sum_{k,j} \delta_k \Bigg\{ (1-N^j_t) \Bigg(\left(a_{jk}e^{-Y_t} + \mu^f_{jk}(t,Z_t) \right) dt + \sum_i \sigma^f_{jki}(t,Z_t) dW_i(t) \Bigg) \\ &- g^{jk}(t,Z_{t-})dN^j_t + \sum_o \left[f^{jk}(t, Z_{t-} + \xi_{j,o}\mathbf{e}_{\lambda_\mathbf{j}}) - f^{jk}(t,Z_{t-}) \right] dN^o_t \Bigg\}. \end{aligned} \tag{39}$$

Consider the augmented vector Markov process Z. The individual components of the drift coefficient $\mu(t,Z_t)$ can be expressed as

$$\begin{aligned}\mu^{x_i}(t,Z_t) &= \kappa_{X_i}(\theta_{X_i} - X_i(t)) \\ \mu^{y}(t,Z_t) &= R(X_t) \\ \mu^{l_k}(t,Z_t) &= L^k(X_t) \\ \mu^{\lambda_j}(t,Z_t) &= \kappa_j(\theta_j - \lambda_j(t)) \\ \mu^{n_i}(t,Z_t) &= 0 \\ \mu^{v}(t,Z_t) &= \sum_{k,j} \delta_k (1-N^j_t) \left(a_{jk}e^{-Y_t} + \mu^f_{jk}(t,Z_t) \right).\end{aligned}$$

Similarly, the rows of the diffusive volatility matrix are given as

$$\begin{aligned}\sigma^{x_i}(t,Z_t) &= \mathbf{e_i} \cdot \sigma_{X_i} \sqrt{X_i(t)} \\ \sigma^{y}(t,Z_t) &= \sigma^{l_k}(t,Z_t) = \sigma^{\lambda_j}(t,Z_t) = \sigma^{n_i}(t,Z_t) = \mathbf{0} \\ \sigma^{v}(t,Z_t) &= \left(\sum_{k,j} \delta_k (1-N^j_t) \sigma_{X_i} \sqrt{X_i(t)} f^{jk}_{x_i} \right)_i.\end{aligned}$$

Since the jumps in V_t only come from jumps in N_t, and the impact coefficients $\xi^{j,l}$ are constant, the jump sizes are completely determined by (t, X_t, Y_t, L_t). This simplifies the expansions. In accordance with the notation of Appendix B, let

$$G^{jk}(t,Z_{t-}) = \sum_o \left[f^{jk}(t, Z_{t-} + \xi_{j,o}\mathbf{e}_{\lambda_\mathbf{j}}) - f^{jk}(t,Z_{t-}) \right] - g^{jk}(t,Z_{t-}).$$

The first and second order expansion results are given by Proposition 3.2 where, suppressing the

arguments of the coefficient functions, we have

$$
\begin{aligned}
c_1 &= \mu^v u + \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv} u^2 + \sum_j \lambda_j(1-n_j)\left(\exp\left(u\sum_k \delta_k G^{jk}\right) - 1\right) \\
2c_2 &= \mu_t^v u + \frac{1}{2}\left[\sigma\sigma^\top\right]_t^{vv} u^2 + \sum_j \lambda_j(1-n_j) u \sum_k \delta_k G_t^{jk} e^{u\sum_m \delta_m G^{j,m}} \\
&+ \left\{\mu_z^v u + \frac{1}{2}\left[\sigma\sigma^\top\right]_z^{vv} u^2 + \left[\sum_j \lambda_j(1-n_j)\left(e^{u\sum_k \delta_k G^{jk}} - 1\right)\right]_z\right\}\mu \\
&+ \left\{\mu^v u^2 + \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv} u^3 + u\sum_j \lambda_j(1-n_j)\left(e^{u\sum_k \delta_k G^{jk}} - 1\right)\right\}\mu^v \\
&+ \frac{1}{2}\mathrm{tr}\left[\left\{\mu_{zz}^v u + \frac{1}{2}\left[\sigma\sigma^\top\right]_{zz}^{vv} u^2 + \left[\sum_j \lambda_j(1-n_j)\left(e^{u\sum_k \delta_k G^{jk}} - 1\right)\right]_{zz}\right\}\sigma\sigma^\top\right] \\
&+ \left\{\mu_z^v u^2 + \frac{1}{2}\left[\sigma\sigma^\top\right]_z^{vv} u^3 + u\left[\sum_j \lambda_j(1-n_j)\left(e^{u\sum_k \delta_k G^{jk}} - 1\right)\right]_z\right\}\left[\sigma\sigma^\top\right]^{\cdot z} \\
&+ \frac{1}{2}\left\{\mu^v u^3 + \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv} u^4 + u\sum_j \lambda_j(1-n_j)\left(e^{u\sum_k \delta_k G^{jk}} - 1\right)\right\}\left[\sigma\sigma^\top\right]^{vv} \\
&+ \sum_j \lambda_j(1-n_j)\left\{e^{u\sum_k \delta_k G^{jk}}\left(\mu_{w_{j+}}^v u + \frac{1}{2}\left[\sigma_{w_j+}\sigma_{w_{j+}}^\top\right]^{vv} u^2 + \sum_{l\neq j}(\lambda_l + \xi_{l,j}(t))(1-n_l)\right.\right. \\
&\qquad \left.\left.\times\left(e^{u\sum_k \delta_k G_{w_{j+}}^{l,k}} - 1\right)\right) - \mu^v u - \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv} u^2 - \sum_l \lambda_l(1-n_l)\left(e^{u\sum_k \delta_k G^{l,k}} - 1\right)\right\}
\end{aligned}
$$

where $\mu_{w_j+} = \mu(t,x,y,\lambda+\xi_{j,\cdot},n+\mathbf{e_j})$, and $\sigma_{w_j+} = \sigma(t,x,y,\lambda+\xi_{j,\cdot},n+\mathbf{e_j})$.

5 Numerical results

This section provides numerical results. We analyze a portfolio of 50 relatively risky loans, 15 of which are hedged through credit swap protection buying positions. Each loan has a principal of \$10 million. We estimate the distribution of the value V_Δ of the hedged portfolio for several horizons Δ, and the corresponding value at risk (VaR) for each of several confidence levels.

The analysis is based on the doubly-stochastic model of default timing described in Section 4.1, with $K = 2$ common risk factors driving default intensities and $K_r = 2$ independent short-rate factors. Each name has a single idiosyncratic factor. To accommodate a potential valuation discrepancy between a loan and a credit swap hedge, we introduce an additional Feller diffusion discount rate process (see Section 2.1). All reversion speed parameters were set to $\kappa = 0.20$. The common risk factor volatilities were set to $\sigma_1 = 0.06$ and $\sigma_2 = 0.15$, and the long-term means were set to $\theta_1 = 0.02$ and $\theta_2 = 0.09$. The short-rate volatilities were set to $\sigma_{r1} = 0.05$ and $\sigma_{r2} = 0.10$. The long-term means of the short rate factors were set to $\theta_{r1} = 0.02$ and $\theta_{r2} = 0.07$. The long-term means

	1Y Default Probability	5Y CDS Spread	Additional Loan Discount
Range	$[0.08\%, 22.15\%]$	$[31.3\text{bp}, 2748.4\text{bp}]$	$[13.3\text{bp}, 424.7\text{bp}]$
Mean	2.87%	291.4bp	95.2bp
Median	2.33%	243.2bp	77.1bp

Table 1: Characteristics of the hedged portfolio: one-year probability of default, 5-year CDS spreads, and (annualized) additional discounts for loan positions.

θ_{X_j} of the idiosyncratic factors were drawn from a uniform distribution on $(0.005, 0.3)$ such that the Feller condition was preserved. The idiosyncratic factor volatilities are automatically determined by the factor loadings, which were drawn from a uniform distribution on $(0.2, 2.5)$. The long term means of the additional loan discount factors were drawn from a uniform distribution on $(0.001, 0.07)$. The volatilities were drawn from $(0.02, 0.15)$, while also checking Feller conditions.

The parameter choices represent a relatively risky "high yield" portfolio. Table 1 summarizes key characteristics of the hedged portfolio. For comparison, the constituent credits of the North American CDX High Yield index had an average one-year default probability of 2.73% on the close of business day 03/31/2011. Since we focus on short horizons, the idiosyncratic θ_{X_j}'s were intentionally drawn from a wide upper range. The goal was to include names with high enough default probability to induce default losses at the left tail of the distribution even for short horizons.

To evaluate the accuracy of the approximation at various Δ, we also estimate the distribution of the value of the hedged portfolio using Monte Carlo simulation. Here we simulate default times in $[0, \Delta]$ based on a standard time-scaling scheme. This scheme requires the discretization of Feller-diffusion intensity processes. We use a daily discretization grid, and generate the intensity values by exact sampling from the non-central chi-squared distribution. If a borrower survives to Δ, then we compute the value of the corresponding position using the analytical expression for f^{jk} available in the formulation of Section 4.1. This eliminates the need of a nested simulation procedure, and allows us to avoid the bias introduced by such a procedure. Nevertheless, some bias remains due to the discretization of the intensity when simulating default times. While it is difficult to quantify the magnitude of this bias (see Giesecke & Teng (2012) in this regard), we expect it to be small for relatively short horizons Δ. The numerical integrations for the compensator (integrated intensity), short-rate, and payoff integrals also follow a daily discretization grid.

The simulation results are based on 75,000 independent Monte Carlo trials. The computational effort required by the simulation is substantial. This is due to:

- The large number of risk factors. In this experiment there are 2 short rate factors, 50 additional loan discount factors, 50 idiosyncratic default intensity factors, 2 common intensity factors, 2 common stochastic measure change factors, and 50 idiosyncratic measure change factors for a total of 156 independent risk factors following Feller diffusion dynamics.

- Path dependency. Although we are only interested in the terminal distribution of the portfolio V_Δ, the compensator, short-rate, and liquidity discount integrals of expression (5) have built-in path dependency. In particular, the payoffs depend on the path of the multivariate default arrival process. This limits our ability to take advantage of fully vectorizable schemes.

	1D	1W	2W	1M
Monte Carlo	3775.2s	22404.7s	58709.7s	137724.4s
Approximation	103.7s	334.9s	539.4s	766.1s

Table 2: Comparison of run-times: Monte Carlo vs. approximation.

- Grid size. In order to reduce the discretization bias, a fine time grid must be used.

Table 2 compares the runtimes of Monte Carlo simulation and approximation. The times are based on a Matlab 7.9.0.529 implementation, running on a desktop computer with Intel Xeon E5620 2.4GHz processor and 3.5GB of RAM. Even for this portfolio of 50 loans and 15 credit swaps, the one-month horizon simulation runtime already nears two full days. The approximation requires much less computation time.

We contrast our approximation results for the distribution of V_Δ with 95% confidence intervals (CIs) for the simulated distribution. We prefer this CI criterion over an alternative relative error criterion based on Monte Carlo point estimates of the distribution of V_Δ. The relative error criterion does not take into account the magnitude of the Monte Carlo noise. Also relative error is a rather arbitrary criterion when there are two competing effects (positive interest income, and negative default payouts) being modeled, as the base quantity can be close to zero, by virtue of two large canceling effects. Suppose the "true" quantile for the positive interest income is 1000, and default payouts 995. Now suppose the approximation yields 1000 and 990, respectively. Then the relative error stands at 100%.

Table 3 reports the approximate 90%, 92.5%, 95%, 97.5% and 99% VaR (left quantiles) of V_Δ, along with the Monte Carlo CIs of the VaR. The approximation results are based on a second-order approximation to the transform of V_Δ and a saddlepoint expansion up to the leading term. Consistent with (21), higher-order saddlepoint expansions were not found to be necessary; we have confirmed the accuracy of the saddlepoint expansions using numerical transform inversion. To estimate the Monte Carlo CIs, we use the bias-corrected acceleration (BCA) bootstrap method. For short horizons of Δ equal to one day and one week, the approximated VaR lies inside the CIs for all confidence levels. For two weeks, only the approximated 99% VaR lies outside of the confidence interval. As expected, due to the nature of the approximation, the accuracy in the tail deteriorates for the longer horizon of one month.

Note that the Monte Carlo CIs contain discretization bias of unknown magnitude. The accuracy trend is, however, consistent with that of alternative approximation methods, including the delta-gamma method. Over horizons longer than several weeks, the accuracy of the delta-gamma method also deteriorates, as larger price moves are more likely. The advantage of our small-time approximation over the delta-gamma approximation is that it is designed to capture important features of the underlying stochastic movements, including nonlinear cash flows, jump events, and stochastic volatility. These features are harder to address using the delta-gamma method (see Duffie & Pan (2001) for a discussion).

	90% VaR		92.5% VaR		95% VaR	
Δ	95% MC CI	Approx.	95% MC CI	Approx.	95% MC CI	Approx.
1D	[1.72872, 1.75604]	1.75551	[3.18809, 3.22567]	3.21579	[4.43649, 4.48972]	4.45380
1W	[2.30661, 2.33114]	2.33046	[3.25952, 3.30137]	3.30044	[5.46249, 5.52178]	5.50813
2W	[4.56686, 4.60444]	4.60013	[7.94990, 8.00747]	7.95667	[11.97272, 12.07023]	11.98025
1M	[5.08102, 5.11867]	5.10223	[8.08840, 8.13472]	8.13983	[13.19563, 13.29322]	13.01512
	97.5% VaR		99% VaR			
Δ	95% MC CI	Approx.	95% MC CI	Approx.		
1D	[6.72876, 6.78824]	6.74963	[9.98898, 10.10359]	10.04428		
1W	[9.43825, 9.50450]	9.49417	[15.16620, 15.29385]	15.17002		
2W	[19.22699, 19.37683]	19.24396	[29.52758, 29.76684]	29.34937		
1M	[22.06711, 22.19805]	20.67538	[34.55052, 34.86024]	29.62684		

Table 3: VaR estimates obtained using the approximation method and 95% confidence interval of VaR estimates obtained using Monte Carlo simulation. Units: $ million.

6 Conclusion

This paper develops an analytical approximation to the probability distribution of the future value of a portfolio of loans and credit derivatives. The approximation is based on a small-time expansion of a transform of the portfolio value, and an estimation of the corresponding distribution using the saddlepoint method. Its scope includes many standard reduced-form models of firm-by-firm default timing. Numerical experiments illustrate the accuracy and computational efficiency of the approximation for a hedged portfolio of loans.

The approximation has a number of applications. It facilitates the design of credit derivatives hedges for corporate loan portfolios, the estimation of risk measures and capital reserves for hedged loan portfolios, and the estimation of risk measure sensitivities. The approximation also facilitates applications in credit derivatives trading and clearing, including the computation of margin requirements for a portfolio of credit swap positions.

A Proofs

Proof of Proposition 3.1. Suppose that the space-time process (t, Z_t^X) is a strong Markov process. Consider the process $\tilde{Z}_t = (X_t, Y_t, \Lambda_t, N_t, \tilde{V}_t)$ where

$$\tilde{V}_t \;=\; \sum_{k,j} \delta_k \Big\{ -v_0^{jk} + a_{jk} \int_0^t (1 - N_v^j) e^{-Y_v} dv - b_{jk} \int_0^t e^{-Y_v} dN_v^j \Big\}.$$

Since e^{-x} has a bounded derivative on the nonnegative interval, it is an autonomous Lipschitz continuous function on $[0, \infty)$. An application Theorem V.35 of Protter (2004) then shows that $(t, \tilde{Z}_t)$ is a strong Markov process. Now we define the vector-valued function H as

$$H(t, \tilde{Z}_t) \;=\; \left(t, Z_t^X, \tilde{V}_t + \sum_{k,j} \delta_k (1 - N_t^j) f^{jk}(t, X_t, Y_t, L_t, \Lambda_t) \right)$$

$$= \quad (t, \tilde{Z}_t) + \left(0, 0, \ldots, 0, \sum_{k,j} \delta_k (1 - N_t^j) f^{jk}(t, X_t, Y_t, L_t, \Lambda_t)\right).$$

By another application of Theorem V.35 of Protter (2004), the functions f^{jk} represent $\mathscr{F}_t$-conditional expectations completely determined by the vector Z_t^X. Iterative conditioning arguments lead to the conclusion that the space-time process $(t, Z_t) = H(t, \tilde{Z}_t)$ is also a strong Markov process.

Next let us consider the univariate process V_t alone. We have

$$|(1 - N_t^j) f^{jk}(t, X_t, Y_t, L_t, \Lambda_t)| \quad \leq \quad a_{jk}(T - t) + b_{jk} + v_T^{jk}$$

which leads to

$$\begin{aligned}
|V_t| &= \left| \sum_{k,j} \delta_k \left\{ - v_0^{jk} + a_{jk} \int_0^t (1 - N_v^j) e^{-Y_v} dv - b_{jk} \int_0^t e^{-Y_v} dN_v^j + (1 - N_t^j) f^{jk}(t, X_t, Y_t, \Lambda_t) \right\} \right| \\
&\leq \sum_{k,j} |\delta_k| \left\{ v_0^{jk} + a_{jk} t + b_{jk} + a_{jk}(T - t) + b_{jk} + v_T^{jk} \right\} \\
&= \sum_{k,j} |\delta_k| \left\{ v_0^{jk} + a_{jk} + 2b_{jk} + v_T^{jk} \right\},
\end{aligned}$$

and therefore $|V_t|$ is a bounded process. Let $\bar{v} = \sum_{k,j} |\delta_k| \{v_0^{jk} + a_{jk} + 2b_{jk} + v_T^{jk}\}$, then the function e^{uv} and all of its subsequent derivatives are bounded and continuous on the compact support $v \in [-\bar{v}, \bar{v}]$, and therefore are in the domain of the infinitesimal generator. The form of the generator is given by an application of Dynkin's formula. □

Proof of Proposition 3.2. The first order expansion of the Laplace transform is given by

$$\begin{aligned}
\psi^{(1)}(\Delta, u|z) = e^{uv} \Bigg(1 + \Bigg(\mu^v u + \frac{1}{2} \left[\sigma\sigma^\top\right]^{vv} u^2 + \sum_m \iota_m \left(\int_{\mathbb{R}} \exp\left(u \sum_{k,j} \delta_k F_J^{jk}(\epsilon) \right) d\nu_m(\epsilon) - 1 \right) \\
+ \sum_j \lambda_j (1 - n_j) \left(\exp\left(u \sum_k \delta_k G^{jk} \right) - 1 \right) \Bigg) \Delta \Bigg)
\end{aligned}$$

where the v and vv superscripts denote the v, and (v, v) components of the respective vectors and matrices. For the Δ-order expansion of the cumulant, we take the log of the above:

$$\begin{aligned}
K^{(1)}(\Delta, u|z) &= uv + \frac{\partial}{\partial \Delta} K(\Delta, u|z) \Big|_{\Delta=0} \Delta \\
&= uv + \left(\mu^v u + \frac{1}{2} \left[\sigma\sigma^\top\right]^{vv} u^2 + \sum_m \iota_m \left(\int_{\mathbb{R}} \exp\left(u \sum_{k,j} \delta_k F^{jk}(\epsilon) \right) d\nu_m(\epsilon) - 1 \right) \right. \\
&\quad \left. + \sum_j \lambda_j (1 - n_j) \left(\exp\left(u \sum_k \delta_k G^{jk} \right) - 1 \right) \right) \Delta.
\end{aligned}$$

The functions F^{jk} and G^{jk} are defined in Appendix B below. The second order expansion in Δ of ψ is given by

$$\psi^{(2)}(\Delta, u|z) = \exp\left(K^{(1)}(\Delta, u|z)\right) \cdot \left(1 + \frac{1}{2}\frac{\partial^2}{\partial \Delta^2} K(\Delta, u|z)\big|_{\Delta=0} \Delta^2\right).$$

The second order cumulant transform $K^{(2)}(\Delta, u|z)$ can be found by taking the Taylor series of

$$\log\left(h(0,z) + \mathscr{A}h(0,z)\Delta + \frac{1}{2}\mathscr{A}^2 h(0,z)\Delta^2\right)$$

up to the second power in Δ around $\Delta = 0$. We have

$$K^{(2)}(\Delta, u|z) = uv + c_1\Delta + \left(-\frac{c_1^2}{2} + c_2\right)\Delta^2 + O(\Delta^3) \tag{40}$$

where

$$\begin{aligned} c_1 &= \frac{\mathscr{A}e^{uv}}{e^{uv}} \\ c_2 &= \frac{1}{2}\frac{\mathscr{A}^2 e^{uv}}{e^{uv}}. \end{aligned}$$

Explicit expressions for c_1 and c_2 are developed in Appendix C below. □

B Jump size vectors

The jump size vectors in the infinitesimal generator expressions are given by

$$\varsigma_m(t,z,\epsilon) = \begin{cases} 0, & \text{for all } x, y, l, \text{ and } n \text{ components} \\ \epsilon_j, & \text{for all } \lambda_j \\ \sum_{k,j} \delta_k F^{jk}(t,x,y,l,\lambda;\epsilon), & \text{for } v, \end{cases}$$

$$w_j(t,z) = \begin{cases} 0, & \text{for all } x, y, \text{ and } l \text{ components, } \lambda_j, \text{ and } n_o \text{ where } o \neq j \\ \xi_{j,o}(t), & \text{for } \lambda_o \text{ where } o \neq j \\ 1, & \text{for } n_j \\ \sum_k \delta_k G^{jk}(t,x,y,l,\lambda), & \text{for } v, \end{cases}$$

where

$$\begin{aligned} F^{jk}(t,x,y,l,\lambda;\epsilon) &= (1 - N_t^j)[f^{jk}(t,x,y,l,\lambda + \epsilon_j \mathbf{e_j}) - f^{jk}(t,x,y,l,\lambda)], \\ G^{jk}(t,x,y,l,\lambda) &= \sum_o \left[f^{jk}(t,x,y,l,\lambda + \xi_{j,o}(t)\mathbf{e_j}) - f^{jk}(t,x,y,l,\lambda)\right] - g^{jk}(t,x,y,l,\lambda), \\ g^{jk}(t,x,y,l,\lambda) &= b_{jk}e^{-y} + f^{jk}(t,x,y,l,\lambda). \end{aligned}$$

C Explicit expressions for c_1 and c_2

Here we give explicit expressions for the functions c_1 and c_2 in Proposition 3.2. To ease the presentation, we suppress the (t,z) dependence of the coefficient functions. We have

$$c_1 = \mu^v u + \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv} u^2 + \sum_m \iota_m \left(\int_{\mathbb{R}} e^{u\sum_{k,j}\delta_k F^{jk}(\epsilon)} d\nu_m(\epsilon) - 1 \right) + \sum_j \lambda_j (1-n_j)\left(e^{u\sum_k \delta_k G^{jk}} - 1 \right),$$

and further

$$\begin{aligned}
2c_2 &= \frac{\mathscr{A}^2 e^{uv}}{e^{uv}} \\
&= \mu_t^v u + \frac{1}{2}\left[\sigma\sigma^\top\right]_t^{vv} u^2 + u\sum_m \sum_{k,j} \iota_m \delta_k \int_{\mathbb{R}} F_t^{jk}(\epsilon) e^{u\sum_q \sum_l \delta_q F^{l,q}(\epsilon)} d\nu_m(\epsilon) \\
&+ \sum_j \lambda_j (1-n_j) u \sum_k \delta_k G_t^{jk} e^{u\sum_m \delta_m G^{j,m}} \\
&+ \left\{ \left[\left(\mu^v u + \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv} u^2 + \sum_m \iota_m \left(\int_{\mathbb{R}} e^{u\sum_{k,j}\delta_k F^{jk}(\epsilon)} d\nu_m(\epsilon) - 1 \right) + \sum_j \lambda_j (1-n_j)\left(e^{u\sum_k \delta_k G^{jk}} - 1 \right) \right) e^{uv} \right]_z \mu \right. \\
&\quad \left. + \frac{1}{2}\mathrm{tr}\left[\left[\left(\mu^v u + \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv} u^2 + \sum_m \iota_m \left(\int_{\mathbb{R}} e^{u\sum_{k,j}\delta_k F^{jk}(\epsilon)} d\nu_m(\epsilon) - 1 \right) \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left. + \sum_j \lambda_j (1-n_j)\left(e^{u\sum_k \delta_k G^{jk}} - 1 \right) \right) e^{uv} \right]_{zz} \sigma\sigma^\top \right] \right\} e^{-uv} \\
&+ \sum_m \iota_m \left(\int_{\mathbb{R}} e^{u\sum_{k,j}\delta_k F^{jk}(\epsilon)} \left(\mu_{m+}^v(\epsilon) u + \frac{1}{2}\left[\sigma_{m+}(\epsilon)\sigma_{m+}(\epsilon)^\top\right]^{vv} u^2 \right. \right. \\
&\quad \left. + \sum_n \iota_{n,m+} \left(\int_{\mathbb{R}} e^{u\sum_{k,j}\delta_k F_{n+}^{jk}(\epsilon,\varepsilon)} d\nu_n(\varepsilon) - 1 \right) \right. \\
&\quad \left. \left. + \sum_j (\lambda_j + \epsilon)(1-n_j)\left(e^{u\sum_k \delta_k G_{m+}^{jk}(\epsilon)} - 1 \right) \right) d\nu_m(\epsilon) \right. \\
&\quad \left. - \mu^v u - \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv} u^2 - \sum_n \left(\iota_n \int_{\mathbb{R}} e^{u\sum_{k,j}\delta_k F^{jk}(\epsilon)} d\nu_n(\epsilon) - 1 \right) - \sum_j \lambda_j (1-n_j)\left(e^{u\sum_k \delta_k G^{jk}} - 1 \right) \right) \\
&+ \sum_j \lambda_j (1-n_j) \left(\left(\mu_{w_j+}^v u + \frac{1}{2}\left[\sigma_{w_j+}\sigma_{w_j+}^\top\right]^{vv} u^2 + \sum_m \iota_{m,w_j+} \left(\int_{\mathbb{R}} e^{u\sum_k \sum_l \delta_l F_{w_j+}^{jk}(\epsilon)} d\nu_m(\epsilon) - 1 \right) \right. \right. \\
&\quad \left. \left. + \sum_l (\lambda_l + \xi_{l,j}(t))(1-n_l)\left(e^{u\sum_k \delta_k G_{w_j+}^{l,k}} - 1 \right) \right) e^{u\sum_k \delta_k G^{jk}} \right. \\
&\quad \left. - \mu^v u - \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv} u^2 - \sum_m \iota_m \left(\int_{\mathbb{R}} e^{u\sum_k \sum_l \delta_l F^{jk}(\epsilon)} d\nu_m(\epsilon) - 1 \right) - \sum_l \lambda_l (1-n_l)\left(e^{u\sum_k \delta_k G^{l,k}} - 1 \right) \right) \\
&= \mu_t^v u + \frac{1}{2}\left[\sigma\sigma^\top\right]_t^{vv} u^2 + u\sum_m \sum_{k,j} \iota_m \delta_k \int_{\mathbb{R}} F_t^{jk}(\epsilon) e^{u\sum_q \sum_l \delta_q F^{l,q}(\epsilon)} d\nu_m(\epsilon) + \sum_j \lambda_j (1-n_j) u \sum_k \delta_k G_t^{jk} e^{u\sum_m \delta_m G^{j,m}} \\
&+ \left\{ \mu_z^v u + \frac{1}{2}\left[\sigma\sigma^\top\right]_z^{vv} u^2 + \left[\sum_m \iota_m \left(\int_{\mathbb{R}} e^{u\sum_{k,j}\delta_k F^{jk}(\epsilon)} d\nu_m(\epsilon) - 1 \right) \right]_z + \left[\sum_j \lambda_j (1-n_j)\left(e^{u\sum_k \delta_k G^{jk}} - 1 \right) \right]_z \right\} \mu \\
&+ \left\{ \mu^v u^2 + \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv} u^3 + u\sum_m \iota_m \left(\int_{\mathbb{R}} e^{u\sum_{k,j}\delta_k F^{jk}(\epsilon)} d\nu_m(\epsilon) - 1 \right) + u\sum_j \lambda_j (1-n_j)\left(e^{u\sum_k \delta_k G^{jk}} - 1 \right) \right\} \mu^v
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}\mathrm{tr}\Bigg[\Bigg\{\mu^v_{zz}u + \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv}_{zz}u^2 + \left[\sum_m \iota_m\left(\int_{\mathbb{R}} e^{u\sum_{k,j}\delta_k F^{jk}(\epsilon)}d\nu_m(\epsilon) - 1\right)\right]_{zz} \\
&\qquad + \left[\sum_j \lambda_j(1-n_j)\left(e^{u\sum_k \delta_k G^{jk}} - 1\right)\right]_{zz}\Bigg\}\sigma\sigma^\top\Bigg] \\
&+ \quad \Bigg\{\mu^v_z u^2 + \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv}_z u^3 + u\left[\sum_m \iota_m\left(\int_{\mathbb{R}} e^{u\sum_{k,j}\delta_k F^{jk}(\epsilon)}d\nu_m(\epsilon) - 1\right)\right]_z \\
&\qquad + u\left[\sum_j \lambda_j(1-n_j)\left(e^{u\sum_k \delta_k G^{jk}} - 1\right)\right]_z\Bigg\}\left[\sigma\sigma^\top\right]^{\cdot z} \\
&+ \quad \frac{1}{2}\Bigg\{\mu^v u^3 + \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv}u^4 + u\sum_m \iota_m\left(\int_{\mathbb{R}} e^{u\sum_{k,j}\delta_k F^{jk}(\epsilon)}d\nu_m(\epsilon) - 1\right) + u\sum_j \lambda_j(1-n_j)\left(e^{u\sum_k \delta_k G^{jk}} - 1\right)\Bigg\}\left[\sigma\sigma^\top\right]^{vv} \\
&+ \quad \sum_m \iota_m\Bigg(\int_{\mathbb{R}} e^{u\sum_{k,j}\delta_k F^{jk}(\epsilon)}\Bigg(\mu^v_{m+}(\epsilon)u + \frac{1}{2}\left[\sigma_{m+}(\epsilon)\sigma_{m+}(\epsilon)^\top\right]^{vv}u^2 \\
&\qquad\qquad + \sum_n \iota_{n,m+}\left(\int_{\mathbb{R}} e^{u\sum_{k,j}\delta_k F^{jk}_{n+}(\epsilon,\varepsilon)}d\nu_n(\varepsilon) - 1\right) \\
&\qquad\qquad + \sum_j (\lambda_j + \epsilon)(1-n_j)\left(e^{u\sum_k \delta_k G^{jk}_{m+}(\epsilon)} - 1\right)\Bigg)d\nu_m(\epsilon) \\
&\qquad - \mu^v u - \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv}u^2 - \sum_n\left(\iota_n\int_{\mathbb{R}} e^{u\sum_{k,j}\delta_k F^{jk}(\epsilon)}d\nu_n(\epsilon) - 1\right) - \sum_j \lambda_j(1-n_j)\left(e^{u\sum_k \delta_k G^{jk}} - 1\right)\Bigg) \\
&+ \quad \sum_j \lambda_j(1-n_j)\Bigg(\Bigg(\mu^v_{w_j+}u + \frac{1}{2}\left[\sigma_{w_j+}\sigma^\top_{w_j+}\right]^{vv}u^2 + \sum_m \iota_{m,w_j+}\left(\int_{\mathbb{R}} e^{u\sum_k\sum_l \delta_l F^{jk}_{w_j+}(\epsilon)}d\nu_m(\epsilon) - 1\right) \\
&\qquad\qquad + \sum_l (\lambda_l + \xi_{l,j}(t))(1-n_l)\left(e^{u\sum_k \delta_k G^{l,k}_{w_j+}} - 1\right)\Bigg)e^{u\sum_k \delta_k G^{jk}} \\
&\qquad\qquad - \mu^v u - \frac{1}{2}\left[\sigma\sigma^\top\right]^{vv}u^2 - \sum_m \iota_m\left(\int_{\mathbb{R}} e^{u\sum_k\sum_l \delta_l F^{jk}(\epsilon)}d\nu_m(\epsilon) - 1\right) - \sum_l \lambda_l(1-n_l)\left(e^{u\sum_k \delta_k G^{l,k}} - 1\right)\Bigg)
\end{aligned}
$$

with, letting $\xi_{j,j} = 0$,

$$
\begin{aligned}
\mu_{m+}(\epsilon) &= \mu(t,x,y,\lambda+\epsilon,n) \\
\sigma_{m+}(\epsilon) &= \sigma(t,x,y,\lambda+\epsilon,n) \\
\iota_{n,m+}(\epsilon) &= \iota_n(\lambda+\epsilon) \\
F^{jk}_{n+}(\epsilon,\varepsilon) &= F^{jk}(t,x,y,\lambda+\varepsilon;\epsilon) \\
G^{jk}_{m+}(\epsilon) &= G^{jk}(t,x,y,\lambda+\epsilon) \\
\mu_{w_j+} &= \mu(t,x,y,\lambda+\xi_{j,\cdot},n+\mathbf{e_j}) \\
\sigma_{w_j+} &= \sigma(t,x,y,\lambda+\xi_{j,\cdot},n+\mathbf{e_j}) \\
\iota_{m,w_j+} &= \iota_m(\lambda+\xi_{j,\cdot}) \\
F^{jk}_{w_j+}(\epsilon) &= F^{jk}(t,x,y,\lambda+\xi_{j,\cdot};\epsilon) \\
G^{jk}_{m+} &= G^{jk}(t,x,y,\lambda+\xi_{j,\cdot}).
\end{aligned}
$$

D Additional calculations in the doubly-stochastic model

Following the approach in Giesecke & Kim (2010), f^{jk} and its derivatives can be computed explicitly by exploiting the doubly stochastic property. We have

$$\begin{aligned}
&\left(a_{jk} \int_t^T \mathbb{E}_t^* \left[e^{-(Y_v - Y_t)} \right] \mathbb{E}_t^* \left[e^{-\int_t^v \lambda_v^{j*} dv} \right] dv - b_{jk} \int_t^T \mathbb{E}_t^* \left[e^{-(Y_v - Y_t)} \right] \left(1 - \mathbb{E}_t^* \left[e^{-\int_t^v \lambda_w^{j*} dw} \right] \right) dv \right. \\
&\left. + (v_T^{jk} - b_{jk}) \mathbb{E}_t^* \left[e^{-(Y_T - Y_t)} \right] \left(1 - \mathbb{E}_t^* \left[e^{-\int_t^T \lambda_w^{j*} dw} \right] \right) \right) \mathbf{1}_{\{\tau_j > t\}} \\
= &\left(a_{jk} \int_t^T \prod_{i \in R,L,I_j} \Psi^{i*}(t, v; \varpi_i, 0) dv - b_{jk} \int_t^T \prod_{i \in R,L} \Psi^{i*}(t, v; \varpi_i, 0) \left(1 - \prod_{i \in I_j} \Psi^{i*}(t, v; \varpi_i, 0) \right) dv \right. \\
&\left. + (v_T^{jk} - b_{jk}) \prod_{i \in R,L} \Psi^{i*}(t, T; \varpi_i, 0) \left(1 - \prod_{i \in I_j} \Psi^{i*}(t, T; \varpi_i, 0) \right) \right) \mathbf{1}_{\{\tau_j > t\}},
\end{aligned}$$

and therefore $f^{jk}(t, X_t, Y_t)$ is given as

$$\begin{aligned}
&f^{jk}(t, X_t, Y_t) \\
= &\left(a_{jk} \int_t^T \prod_{i \in R,L,I_j} \Psi^{i*}(t, v; \varpi_i, 0) dv - b_{jk} \int_t^T \prod_{i \in R,L} \Psi^{i*}(t, v; \varpi_i, 0) \left(1 - \prod_{i \in I_j} \Psi^{i*}(t, v; \varpi_i, 0) \right) dv \right. \\
&\left. + (v_T^{jk} - b_{jk}) \prod_{i \in R,L} \Psi^{i*}(t, T; \varpi_i, 0) \left(1 - \prod_{i \in I_j} \Psi^{i*}(t, T; \varpi_i, 0) \right) \right) e^{-Y_t}
\end{aligned}$$

where R, L, and I_j are the sets of indices for the factors that contribute to the short rate, discount rate, and jth intensity process respectively, and

$$\varpi_i = \begin{cases} r_i & \text{for } i \in R \\ \rho_i & \text{for } i \in L \\ \omega_i^j & \text{for } i \in I_j, \end{cases}$$

represents the corresponding factor loadings. We assume that these three processes are independent. The Ψ functions represent the extended transforms of the corresponding factors

$$\Psi^{i*}(t, s; u, z) = \exp\left(\mu^{i*}(t, s; u, z) + \nu^{i*}(t, s; u, z) X_i(t) \right) \quad \text{for } i \in R, L,$$

and

$$\begin{aligned}
\Psi^{i*}(t, s; u, z) &= \exp\left(\mu^{i*}(t, s; u, z) + \nu^{i*}(t, s; u, z) X_i^*(t) \right) \\
&= \exp\left(\mu^{i*}(t, s; u, z) + \alpha_i \nu^{i*}(t, s; u, z) + \beta_i \nu^{i*}(t, s; u, z) X_i(t) + \nu^{i*}(t, s; u, z) u_i(t) \right)
\end{aligned}$$

for $i \in I_j$, where the functions μ, and ν are the explicit solutions to the system of Riccati ODEs that arise from the extended transform of Feller diffusions as specified Giesecke & Kim (2010). Note that in the last transform we must convert the factor X_i^* into X_i by the relationship in Section 4.1.

Now we can easily compute the relevant derivatives. We have

$$\begin{aligned} f_t^{jk}(t, X_t, Y_t) &= \left\{ a_{jk} \left(-1 + \int_t^T \sum_{i \in R,L,I_j} \frac{\Psi_t^{i*}(t,v;\varpi_i,0)}{\Psi^{i*}(t,v;\varpi_i,0)} \prod_{l \in R,L,I_j} \Psi^{l*}(t,v;\varpi_l,0) dv \right) \right. \\ &- b_{jk} \int_t^T \sum_{i \in R,L} \frac{\Psi_t^{i*}(t,v;\varpi_i,0)}{\Psi^{i*}(t,v;\varpi_i,0)} \prod_{l \in R,L} \Psi^{l*}(t,v;\varpi_l,0) \left(1 - \prod_{l \in I_j} \Psi_X^{l*}(t,v;\varpi_l,0) \right) dv \\ &+ b_{jk} \int_t^T \sum_{i \in I_j} \frac{\Psi_t^{i*}(t,v;\varpi_i,0)}{\Psi^{i*}(t,v;\varpi_i,0)} \prod_{l \in R,L,I_j} \Psi^{l*}(t,v;\varpi_l,0) dv \\ &+ (v_T^{jk} - b_{jk}) \sum_{i \in R,L} \frac{\Psi_t^{i*}(t,T;\varpi_i,0)}{\Psi^{i*}(t,T;\varpi_i,0)} \prod_{l \in R,L} \Psi^{l*}(t,T;\varpi_l,0) \left(1 - \prod_{l \in I_j} \Psi_X^{l*}(t,T;\varpi_l,0) \right) \\ &\left. - (v_T^{jk} - b_{jk}) \sum_{i \in I_j} \frac{\Psi_t^{i*}(t,T;\varpi_i,0)}{\Psi^{i*}(t,T;\varpi_i,0)} \prod_{l \in R,L,I_j} \Psi^{l*}(t,T;\varpi_l,0) \right\} e^{-Y_t} \end{aligned}$$

for the time derivative. Here

$$\frac{\Psi_t^{i*}(t,v;u,z)}{\Psi^{i*}(t,v;u,z)} = \mu_t^{i*}(t,s;u,z) + \nu_t^{i*}(t,s;u,z) X_i(t)$$

for $i \in R, L$, and

$$\frac{\Psi_t^{i*}(t,v;u,z)}{\Psi^{i*}(t,v;u,z)} = \mu_t^{i*}(t,s;u,z) + \alpha_i \nu_t^{i*}(t,s;u,z) + \beta_i \nu_t^{i*}(t,s;u,z) X_i(t) + \nu_t^{i*}(t,s;u,z) u_i(t)$$

for $i \in I_j$. The y-derivative is simply

$$f_y^{jk}(t, X_t, Y_t) = -f^{jk}(t, X_t, Y_t),$$

while

$$\begin{aligned} f_{x_i}^{jk}(t, X_t, Y_t) &= \left\{ a_{jk} \int_t^T \nu^{i*}(t,v;\varpi_i,0) \prod_{l \in R,L,I_j} \Psi^{l*}(t,v;\varpi_l,0) dv \right. \\ &- b_{jk} \int_t^T \nu^{i*}(t,v;\varpi_i,0) \prod_{l \in R,L} \Psi^{l*}(t,v;\varpi_l,0) \left(1 - \prod_{l \in I_j} \Psi_X^{l*}(t,v;\varpi_l,0) \right) dv \\ &\left. + (v_T^{jk} - b_{jk}) \nu^{i*}(t,T;\varpi_i,0) \prod_{l \in R,L} \Psi^{l*}(t,T;\varpi_l,0) \left(1 - \prod_{l \in I_j} \Psi_X^{l*}(t,T;\varpi_l,0) \right) \right\} e^{-Y_t} \end{aligned}$$

for $i \in R, L$,

$$\begin{aligned} f^{jk}_{x_i}(t, X_t, Y_t) &= \Bigg\{ a_{jk} \int_t^T \beta_i \nu^{i*}(t, v; \varpi_i, 0) \prod_{l \in R,L,I_j} \Psi^{l*}(t, v; \varpi_l, 0) dv \\ &\quad + b_{jk} \int_t^T \beta_i \nu^{i*}(t, v; \varpi_i, 0) \prod_{l \in R,L,I_j} \Psi^{l*}_X(t, v; \varpi_l, 0) dv \\ &\quad - (v^{jk}_T - b_{jk}) \beta_i \nu^{i*}(t, T; \varpi_i, 0) \prod_{l \in R,L,I_j} \Psi^{l*}_X(t, T; \varpi_l, 0) \Bigg\} e^{-Y_t} \end{aligned}$$

for $i \in J$, and

$$\begin{aligned} f^{jk}_{u_i}(t, X_t, Y_t) &= \Bigg\{ a_{jk} \int_t^T \nu^{j*}(t, v; \varpi_j, 0) \prod_{l \in R,L,I_j} \Psi^{l*}(t, v; \varpi_l, 0) dv \\ &\quad + b_{jk} \int_t^T \nu^{j*}(t, v; \varpi_j, 0) \prod_{l \in R,L,I_j} \Psi^{l*}_X(t, v; \varpi_l, 0) dv \\ &\quad - (v^{jk}_T - b_{jk}) \nu^{j*}(t, T; \varpi_j, 0) \prod_{l \in R,L,I_j} \Psi^{l*}_X(t, T; \varpi_l, 0) \Bigg\} e^{-Y_t}. \end{aligned}$$

In the application of Itô's formula for computing μ^f_{jk}, the cross derivative terms vanish by the independence of the Brownian motions. Furthermore we do not need f^{jk}_{yy} as the dynamics of the process Y_t does not have a diffusion term. The other second derivative terms are given by

$$\begin{aligned} f^{jk}_{x_i x_i}(t, X_t, Y_t) &= \Bigg\{ a_{jk} \int_t^T \nu^{i*}(t, v; \varpi_i, 0)^2 \prod_{l \in R,L,I_j} \Psi^{l*}(t, v; \varpi_l, 0) dv \\ &\quad - b_{jk} \int_t^T \nu^{i*}(t, v; \varpi_i, 0)^2 \prod_{l \in R,L} \Psi^{l*}(t, v; \varpi_l, 0) \left(1 - \prod_{l \in I_j} \Psi^{l*}_X(t, v; \varpi_l, 0) \right) dv \\ &\quad + (v^{jk}_T - b_{jk}) \nu^{i*}(t, T; \varpi_i, 0)^2 \prod_{l \in R,L} \Psi^{l*}(t, T; \varpi_l, 0) \left(1 - \prod_{l \in I_j} \Psi^{l*}_X(t, T; \varpi_l, 0) \right) \Bigg\} e^{-Y_t} \end{aligned}$$

for $i \in R, L$,

$$\begin{aligned} f^{jk}_{x_i x_i}(t, X_t, Y_t) &= \Bigg\{ a_{jk} \int_t^T \beta_i^2 \nu^{i*}(t, v; \varpi_i, 0)^2 \prod_{l \in R,L,I_j} \Psi^{l*}(t, v; \varpi_l, 0) dv \\ &\quad + b_{jk} \int_t^T \beta_i^2 \nu^{i*}(t, v; \varpi_i, 0)^2 \prod_{l \in R,L,I_j} \Psi^{l*}_X(t, v; \varpi_l, 0) dv \end{aligned}$$

$$- (v_T^{jk} - b_{jk})\beta_i^2 \nu^{i*}(t,T;\varpi_i,0)^2 \prod_{l\in R,L,I_j} \Psi_X^{l*}(t,T;\varpi_l,0)\Bigg\} e^{-Y_t}$$

for $i \in J$, and

$$\begin{aligned} f^{jk}_{u_i u_i}(t,X_t,Y_t) &= \Bigg\{ a_{jk}\int_t^T \nu^{j*}(t,v;\varpi_j,0)^2 \prod_{l\in R,L,I_j} \Psi^{l*}(t,v;\varpi_l,0)dv \\ &+ b_{jk}\int_t^T \nu^{j*}(t,v;\varpi_j,0)^2 \prod_{l\in R,L,I_j} \Psi_X^{l*}(t,v;\varpi_l,0)dv \\ &- (v_T^{jk} - b_{jk})\nu^{j*}(t,T;\varpi_j,0)^2 \prod_{l\in R,L,I_j} \Psi_X^{l*}(t,T;\varpi_l,0)\Bigg\} e^{-Y_t}. \end{aligned}$$

These terms now completely specify μ^v and σ^v.

For completeness we also include the details of the function μ, and ν. These time dependent coefficient functions and their first and second derivatives are given as

$$\begin{aligned} \mu(t,s;u,z) &= DB(t-s) + \frac{2DC}{\gamma}\log\frac{\cosh\left(0.5\gamma(t-s)+A\right)}{\cosh(A)} \\ \nu(t,s;u,z) &= \frac{1}{\sigma^2}\left(\kappa + \gamma\tanh\left(0.5\gamma(t-s)+A\right)\right) \\ \mu_t(t,s;u,z) &= DB + DC\tanh\left(0.5\gamma(t-s)+A\right) \\ \nu_t(t,s;u,z) &= \frac{\gamma^2}{2\sigma^2}\mathrm{sech}\left(0.5\gamma(t-s)+A\right)^2 \\ \mu_{tt}(t,s;u,z) &= \frac{1}{2}DC\gamma\mathrm{sech}\left(0.5\gamma(t-s)+A\right)^2 \\ \nu_{tt}(t,s;u,z) &= -\frac{\gamma^3}{2\sigma^2}\tanh\left(0.5\gamma(t-s)+A\right)\mathrm{sech}\left(0.5\gamma(t-s)+A\right)^2 \end{aligned}$$

where

$$\gamma = \left(\kappa^2 + 2u\sigma^2\right)^{1/2},\ A = \mathrm{atanh}\left(\frac{-\kappa + z\sigma^2}{\gamma}\right),\ B = \frac{\kappa}{\sigma^2},\ C = \frac{\gamma}{\sigma^2},\ D = -\kappa\theta.$$

Several other derivatives are required for the second order terms in the saddlepoint approximation. The computation of these is lengthy and tedious but leads to explicit expressions. The details are available upon request.

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